Exercises on linear transformations and their matrices

Problem 26.1: Consider the transformation T that doubles the distance between each point and the origin without changing the direction from the origin to the points. In polar coordinates this is described by

$$T(r,\theta) = (2r,\theta).$$

a) Yes or no: is T a linear transformation?

b) Describe T using Cartesian (xy) coordinates. Check your work by confirming that the transformation doubles the lengths of vectors.

c) If your answer to (a) was "yes", find the matrix of T. If your answer to (a) was "no", explain why the T isn't linear.

Solution:

a) Yes. In terms of vectors, $T(\mathbf{v}) = 2\mathbf{v}$, so $T(\mathbf{v}_1 + \mathbf{v}_2) = 2\mathbf{v}_1 + 2\mathbf{v}_2 = T(\mathbf{v}_1) + T(\mathbf{v}_2)$ and $T(c\mathbf{v}) = 2c\mathbf{v} = cT(\mathbf{v})$.

b) $T(x,y) = (2x,2y)$. We know $\left|\left| \begin{bmatrix} x \\ y \end{bmatrix} \right|\right| = \sqrt{x^2+y^2}$ and can calculate

$$\left| T\left(\begin{bmatrix} x \\ y \end{bmatrix} \right) \right| = \left|\left| \begin{bmatrix} 2x \\ 2y \end{bmatrix} \right|\right| = \sqrt{4(x^2+y^2)} = 2\left|\left| \begin{bmatrix} x \\ y \end{bmatrix} \right|\right|.$$

This confirms that T doubles the lengths of vectors.

c) We can use our answer to (b) to find that the matrix of T is $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

Problem 26.2: Describe a transformation which leaves the zero vector fixed but which is not a linear transformation.

Solution: If we limit ourselves to "simple" transformations, this is not an easy task!

One way to solve this is to find a transformation that acts differently on different parts of the plane. If $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ |y| \end{bmatrix}$ then

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

is not equal to

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

Another approach is to use a nonlinear function to define the transformation: if $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y^2 \end{bmatrix}$ then $T\left(c\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} cx \\ c^2y^2 \end{bmatrix}$ is not equal to $cT\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} cx \\ cy^2 \end{bmatrix}$.