

Exercises on change of basis; image compression

Problem 26.1: Verify that the vectors of the Haar wavelet basis, given in lecture, are orthogonal. Adjust their lengths so that the resulting basis vectors are orthonormal.

Solution: The vectors given for the Haar wavelet basis were:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

It's easy to see that the second through eighth vectors are orthogonal to the first; their inner products are sums of equal numbers of ones and negative ones. The same thing happens when taking the dot products of the third through eighth vectors with the second. The remaining inner products are either sums of equal numbers of ones and negative ones or simply sums of zeros. Since all the pairwise inner products are zero, the vectors are orthogonal.

To make the basis orthonormal, divide by the lengths of the vectors:

$$\begin{bmatrix} 1/\sqrt{8} \\ 1/\sqrt{8} \\ 1/\sqrt{8} \\ 1/\sqrt{8} \\ 1/\sqrt{8} \\ 1/\sqrt{8} \\ 1/\sqrt{8} \\ 1/\sqrt{8} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{8} \\ 1/\sqrt{8} \\ 1/\sqrt{8} \\ 1/\sqrt{8} \\ -1/\sqrt{8} \\ -1/\sqrt{8} \\ -1/\sqrt{8} \\ -1/\sqrt{8} \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}.$$

Problem 26.2: We can think of the set of all two by two matrices with real valued entries as a vector space. Describe two different bases for this space. Is one of your bases better than the other for describing diagonal matrices? What about triangular matrices? Symmetric matrices?

Solution: There are many different ways to answer this question correctly.

The most obvious choice of basis is:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hopefully, the second basis you chose differs from this one significantly. Here are some possibilities:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

If you have trouble checking that the matrices in your basis are independent, think of them as vectors in \mathbb{R}^4 and check that those vectors are independent. The examples above are easy to check because, when considered as vectors in \mathbb{R}^4 , they are orthogonal.

Of these bases, the first may be best for describing diagonal matrices; the third is also reasonably good for that task:

$$\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} = 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 1 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The first basis also looks best for describing triangular matrices – any triangular matrix is a combination of only three of the four vectors. The third is best for describing symmetric matrices because symmetric matrices will always have a zero coefficient for the $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ component.