## Exercises on left and right inverses; pseudoinverse

Problem 26.1: Find a right inverse for $A=\left[\begin{array}{lll}1 & 0 & 1 \\ 0 & 1 & 0\end{array}\right]$.
Solution: We apply the formula $A_{\text {right }}^{-1}=A^{T}\left(A A^{T}\right)^{-1}$ :

$$
\begin{aligned}
A^{T} & =\left[\begin{array}{ll}
1 & 0 \\
0 & 1 \\
1 & 0
\end{array}\right] \\
A A^{T} & =\left[\begin{array}{ll}
2 & 0 \\
0 & 1
\end{array}\right] \\
\left(A A^{T}\right)^{-1} & =\left[\begin{array}{cc}
1 / 2 & 0 \\
0 & 1
\end{array}\right] \\
A^{T}\left(A A^{T}\right)^{-1} & =\left[\begin{array}{cc}
1 / 2 & 0 \\
0 & 1 \\
1 / 2 & 0
\end{array}\right]
\end{aligned}
$$

Thus, $A_{\text {right }}^{-1}=\left[\begin{array}{cc}1 / 2 & 0 \\ 0 & 1 \\ 1 / 2 & 0\end{array}\right]$ is one right inverse of $A$. Another possible answer is $A_{\text {right }}^{-1}=\left[\begin{array}{ll}1 & 0 \\ 0 & 1 \\ 0 & 0\end{array}\right]$. We can quickly check that $A A_{\text {right }}^{-1}=I$.

Problem 26.2: Does the matrix $A=\left[\begin{array}{ll}4 & 3 \\ 8 & 6\end{array}\right]$ have a left inverse? A right inverse? A pseudoinverse? If the answer to any of these questions is "yes", find the appropriate inverse.

Solution: The second row of $A$ is a multiple of the first row, so $A$ has rank 1 and $\operatorname{det} A=0$. Because $A$ is a square matrix its determinant is defined, and we can use the fact that $\operatorname{det} C \cdot \operatorname{det} D=\operatorname{det}(C D)$ to prove that $A$ can't have a left or right inverse. (If $A B=I$, then $\operatorname{det} A \operatorname{det} B=\operatorname{det} I$ implies $0=1$.)

We can find a pseudoinverse $A^{+}=V \Sigma^{+} U^{T}$ for $A$. We start by finding the singular value decomposition $U \Sigma V^{T}$ of $A$.

The SVD of $A$ was calculated in the lecture on singular value decomposition, so we know that

$$
\underset{A}{\left[\begin{array}{ll}
4 & 3 \\
8 & 6
\end{array}\right]}=\frac{1}{\sqrt{5}} \underset{U}{\left[\begin{array}{lr}
1 & 2 \\
2 & -1
\end{array}\right]} \underset{\Sigma}{\left[\begin{array}{rr}
\sqrt{125} & 0 \\
0 & 0
\end{array}\right]} \underset{V^{T}}{\left[\begin{array}{rr}
.8 & .6 \\
.6 & -.8
\end{array}\right]} .
$$

Hence, $\Sigma^{+}=\left[\begin{array}{rr}1 / \sqrt{125} & 0 \\ 0 & 0\end{array}\right]$ and

$$
\begin{aligned}
A^{+} & =V \Sigma^{+} U^{T} \\
& =\left[\begin{array}{rr}
.8 & .6 \\
.6 & -.8
\end{array}\right]\left[\begin{array}{rr}
1 / \sqrt{125} & 0 \\
0 & 0
\end{array}\right]\left(\frac{1}{\sqrt{5}}\left[\begin{array}{rr}
1 & 2 \\
2 & -1
\end{array}\right]\right) \\
& =\left[\begin{array}{rr}
.8 & .6 \\
.6 & -.8
\end{array}\right]\left[\begin{array}{rr}
1 / 25 & 0 \\
0 & 0
\end{array}\right]\left[\begin{array}{rr}
1 & 2 \\
2 & -1
\end{array}\right] \\
& =\left[\begin{array}{rr}
.8 & .6 \\
.6 & -.8
\end{array}\right]\left[\begin{array}{rr}
1 / 25 & 0 \\
0 & 0
\end{array}\right]\left[\begin{array}{rr}
1 & 2 \\
2 & -1
\end{array}\right] \\
& =\frac{1}{125}\left[\begin{array}{ll}
4 & 8 \\
3 & 6
\end{array}\right] .
\end{aligned}
$$

To check our work, we confirm that $A^{+}$reverses the operation of $A$ on its row space using the bases we found while computing its SVD. Recall that

$$
A \mathbf{v}_{j}= \begin{cases}\sigma_{j} \mathbf{u}_{j} & \text { for } j \leq r \\ \mathbf{0} & \text { for } j>r\end{cases}
$$

Here $\mathbf{u}_{1}=\left[\begin{array}{l}1 / \sqrt{5} \\ 2 / \sqrt{5}\end{array}\right]$ and $A^{+} \mathbf{u}_{1}=\frac{1}{\sqrt{125}}\left[\begin{array}{l}4 \\ 3\end{array}\right]=\frac{1}{\sigma_{1}} \mathbf{v}_{1}$. We can also check that $A^{+} \mathbf{u}_{2}=\frac{1}{125}\left[\begin{array}{ll}4 & 8 \\ 3 & 6\end{array}\right]\left[\begin{array}{r}2 \\ -1\end{array}\right]=\mathbf{0}$.

