Exercises on transposes, permutations, spaces

Problem 5.1: (2.7 #13. *Introduction to Linear Algebra:* Strang)

a) Find a 3 by 3 permutation matrix with $P^3 = I$ (but not $P = I$).

b) Find a 4 by 4 permutation $\widehat{P}$ with $\widehat{P}^4 \neq I$.

Solution:

a) Let P move the rows in a cycle: the first to the second, the second to the third, and the third to the first. So

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, P^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \text{ and } P^3 = I.$$

b) Let $\widehat{P}$ be the block diagonal matrix with 1 and P on the diagonal; $\widehat{P} = \begin{bmatrix} 1 & 0 \\ 0 & P \end{bmatrix}$. Since $P^3 = I$, also $\widehat{P}^3 = I$. So $\widehat{P}^4 = \widehat{P} \neq I$.

Problem 5.2: Suppose A is a four by four matrix. How many entries of A can be chosen independently if:

a) A is symmetric?

b) A is *skew-symmetric*? ($A^T = -A$)

Solution:

a) The most general form of a four by four symmetric matrix is:

$$A = \begin{bmatrix} a & e & f & g \\ e & b & h & i \\ f & h & c & j \\ g & i & j & d \end{bmatrix}.$$

Therefore **10** entries can be chosen independently.

b) The most general form of a four by four skew-symmetric matrix is:

$$A = \begin{bmatrix} 0 & -a & -b & -c \\ a & 0 & -d & -e \\ b & d & 0 & -f \\ c & e & f & 0 \end{bmatrix}.$$

Therefore **6** entries can be chosen independently.

Problem 5.3: (3.1 #18.) True or false (check addition or give a counterexample):

a) The symmetric matrices in M (with $A^T = A$) form a subspace.

b) The skew-symmetric matrices in M (with $A^T = -A$) form a subspace.

c) The unsymmetric matrices in M (with $A^T \neq A$) form a subspace.

Solution:

a) True: $A^T = A$ and $B^T = B$ lead to:

$$(A + B)^T = A^T + B^T = A + B, \text{ and } (cA)^T = cA.$$

b) True: $A^T = -A$ and $B^T = -B$ lead to:

$$(A + B)^T = A^T + B^T = -A - B = -(A + B), \text{ and } (cA)^T = -cA.$$

c) False: $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$