## Exercises on matrix spaces; rank 1; small world graphs

**Problem 11.1:** [Optional] (3.5 #41. *Introduction to Linear Algebra:* Strang) Write the 3 by 3 identity matrix as a combination of the other five permutation matrices. Then show that those five matrices are linearly independent. (Assume a combination gives  $c_1P_1 + \cdots + c_5P_5 = 0$  and check entries to prove  $c_i$  is zero.) The five permutation matrices are a basis for the subspace of three by three matrices with row and column sums all equal.

**Solution:** The other five permutation matrices are:

$$P_{21} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, P_{31} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, P_{32} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, P_{32} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, P_{32} P_{21} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$P_{32}P_{21} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ and } P_{21}P_{32} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Since  $P_{21} + P_{31} + P_{32}$  is the all ones matrix and  $P_{32}P_{21} + P_{21}P_{32}$  is the matrix with zeros on the diagonal and ones elsewhere,

$$I = P_{21} + P_{31} + P_{32} - P_{32}P_{21} - P_{21}P_{32}.$$

For the second part, setting  $c_1P_1 + \cdots + c_5P_5$  equal to zero gives:

$$\begin{bmatrix} c_3 & c_1 + c_4 & c_2 + c_5 \\ c_1 + c_5 & c_2 & c_3 + c_4 \\ c_2 + c_4 & c_3 + c_5 & c_1 \end{bmatrix} = 0.$$

So  $c_1 = c_2 = c_3 = 0$  along the diagonal, and  $c_4 = c_5 = 0$  from the offdiagonal entries.

**Problem 11.2:** (3.6 #31.) **M** is the space of three by three matrices. Multiply each matrix *X* in **M** by:

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$
  
Notice that  $A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ .

- a) Which matrices *X* lead to AX = 0?
- b) Which matrices have the form *AX* for some matrix *X*?
- c) Part (a) finds the "nullspace" of the operation AX and part (b) finds the "column space." What are the dimensions of those two subspaces of **M**? Why do the dimensions add to (n r) + r = 9?

## Solution:

a) We can use row reduction or some other method to see that the rows of *A* are dependent and that *A* has rank 2. Its nullspace has the basis:

$$\left[\begin{array}{c}1\\1\\1\end{array}\right].$$

AX = 0 precisely when the columns of *X* are in the nullspace of *A*, i.e. when they are multiples of the basis of N(A). Therefore, if AX = 0 then *X* must have the form:

$$X = \left[ \begin{array}{rrr} a & b & c \\ a & b & c \\ a & b & c \end{array} \right].$$

b) On the other hand, the columns of any matrix of the form *AX* are linear combinations of the columns of *A*. That is, they are vectors whose components all sum to 0, so a matrix has the form *AX* if and only if all of its columns individually sum to 0:

$$AX = B \text{ if and only if } B = \begin{bmatrix} a & b & c \\ d & e & f \\ -a - d & -b - e & -c - f \end{bmatrix}.$$

c) The dimension of the "nullspace" is 3, while the dimension of the "column space" is 6. These add up to 9, which is the dimension of the space of "inputs" **M**.