## Exercises on matrix spaces; rank 1; small world graphs

Problem 11.1: [Optional] (3.5 \#41. Introduction to Linear Algebra: Strang) Write the 3 by 3 identity matrix as a combination of the other five permutation matrices. Then show that those five matrices are linearly independent. (Assume a combination gives $c_{1} P_{1}+\cdots+c_{5} P_{5}=0$ and check entries to prove $c_{i}$ is zero.) The five permutation matrices are a basis for the subspace of three by three matrices with row and column sums all equal.

Solution: The other five permutation matrices are:

$$
\begin{gathered}
P_{21}=\left[\begin{array}{lll}
1 & 1 & \\
1 & & \\
& & 1
\end{array}\right], P_{31}=\left[\begin{array}{lll} 
& & 1 \\
1 & &
\end{array}\right], P_{32}=\left[\begin{array}{lll}
1 & & \\
& & 1 \\
& 1 &
\end{array}\right], \\
P_{32} P_{21}=\left[\begin{array}{lll} 
& 1 & \\
1 & &
\end{array}\right] \text { and } P_{21} P_{32}=\left[\begin{array}{lll}
1 & & 1 \\
& 1 &
\end{array}\right] .
\end{gathered}
$$

Since $P_{21}+P_{31}+P_{32}$ is the all ones matrix and $P_{32} P_{21}+P_{21} P_{32}$ is the matrix with zeros on the diagonal and ones elsewhere,

$$
I=P_{21}+P_{31}+P_{32}-P_{32} P_{21}-P_{21} P_{32}
$$

For the second part, setting $c_{1} P_{1}+\cdots+c_{5} P_{5}$ equal to zero gives:

$$
\left[\begin{array}{ccc}
c_{3} & c_{1}+c_{4} & c_{2}+c_{5} \\
c_{1}+c_{5} & c_{2} & c_{3}+c_{4} \\
c_{2}+c_{4} & c_{3}+c_{5} & c_{1}
\end{array}\right]=0
$$

So $c_{1}=c_{2}=c_{3}=0$ along the diagonal, and $c_{4}=c_{5}=0$ from the offdiagonal entries.

Problem 11.2: (3.6 \#31.) $\mathbf{M}$ is the space of three by three matrices. Multiply each matrix $X$ in $\mathbf{M}$ by:

$$
A=\left[\begin{array}{rrr}
1 & 0 & -1 \\
-1 & 1 & 0 \\
0 & -1 & 1
\end{array}\right]
$$

Notice that $A\left[\begin{array}{l}1 \\ 1 \\ 1\end{array}\right]=\left[\begin{array}{l}0 \\ 0 \\ 0\end{array}\right]$.
a) Which matrices $X$ lead to $A X=0$ ?
b) Which matrices have the form $A X$ for some matrix $X$ ?
c) Part (a) finds the "nullspace" of the operation $A X$ and part (b) finds the "column space." What are the dimensions of those two subspaces of $\mathbf{M}$ ? Why do the dimensions add to $(n-r)+r=9$ ?

## Solution:

a) We can use row reduction or some other method to see that the rows of $A$ are dependent and that $A$ has rank 2. Its nullspace has the basis:

$$
\left[\begin{array}{l}
1 \\
1 \\
1
\end{array}\right]
$$

$A X=0$ precisely when the columns of $X$ are in the nullspace of $A$, i.e. when they are multiples of the basis of $N(A)$. Therefore, if $A X=0$ then $X$ must have the form:

$$
X=\left[\begin{array}{lll}
a & b & c \\
a & b & c \\
a & b & c
\end{array}\right]
$$

b) On the other hand, the columns of any matrix of the form $A X$ are linear combinations of the columns of $A$. That is, they are vectors whose components all sum to 0 , so a matrix has the form $A X$ if and only if all of its columns individually sum to 0 :

$$
A X=B \text { if and only if } B=\left[\begin{array}{ccc}
a & b & c \\
d & e & f \\
-a-d & -b-e & -c-f
\end{array}\right]
$$

c) The dimension of the "nullspace" is 3, while the dimension of the "column space" is 6 . These add up to 9 , which is the dimension of the space of "inputs" M.

