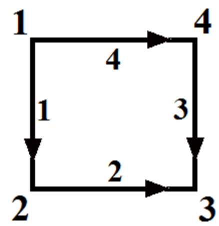
Exercises on graphs, networks, and incidence matrices

Problem 12.1: (8.2 #1. *Introduction to Linear Algebra:* Strang) Write down the four by four incidence matrix A for the square graph, shown below. (Hint: the first row has -1 in column 1 and +1 in column 2.) What vectors (x_1, x_2, x_3, x_4) are in the nullspace of A? How do you know that (1,0,0,0) is not in the row space of A?



Solution: The incidence matrix A is written as:

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix}.$$

To find the vectors in the nullspace, we solve $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_3 - x_4 \\ x_4 - x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

so $x_1 = x_2 = x_3 = x_4$. Therefore, the nullspace consists of vectors of the form (a, a, a, a).

Finally, (1,0,0,0) is not in the row space of A because it is not orthogonal to the nullspace.

Problem 12.2: (8.2 #7.) Continuing with the network from problem one, suppose the conductance matrix is

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Multiply matrices to find A^TCA. For $\mathbf{f} = (1, 0, -1, 0)$, find a solution to $A^TCA\mathbf{x} = \mathbf{f}$. Write the potentials $\mathbf{x}$ and currents $\mathbf{y} = -CA\mathbf{x}$ on the square graph (see above) for this current source $\mathbf{f}$ going into node 1 and out from node 3.

Solution: From the previous question, we know that the incidence matrix is:

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix}.$$

Multiply to obtain A^TCA:

$$\begin{bmatrix} -1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ -1 & 0 & -2 & 3 \end{bmatrix}.$$

Solving the equation $A^TCA\mathbf{x} = \mathbf{f}$ by performing row reduction on the augmented matrix

$$\left[\begin{array}{cccc|c} -1 & 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right]$$

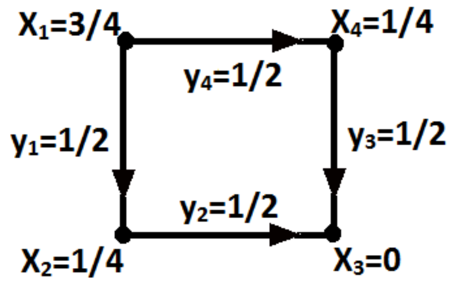
and choosing $x_3 = 0$ to represent a grounded node gives:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \mathbf{3/4} \\ \mathbf{1/4} \\ \mathbf{0} \\ \mathbf{1/4} \end{bmatrix}.$$

We know $\mathbf{y} = -CA\mathbf{x}$, so

$$\mathbf{y} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & -2 & 2 \\ 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3/4 \\ 1/4 \\ 0 \\ 1/4 \end{bmatrix} = \begin{bmatrix} \mathbf{1/2} \\ \mathbf{1/2} \\ \mathbf{1/2} \\ \mathbf{1/2} \end{bmatrix}.$$

We draw these values on the square graph:


X1=3/4
X4=1/4
y4=1/2
y1=1/2
y3=1/2
y2=1/2
X2=1/4
X3=0