Exercises on orthogonal vectors and subspaces

Problem 16.1: (4.1 #7. *Introduction to Linear Algebra:* Strang) For every system of m equations with no solution, there are numbers $y_1, ..., y_m$ that multiply the equations so they add up to $0 = 1$. This is called *Fredholm's Alternative*:

Exactly one of these problems has a solution:
$A\mathbf{x} = \mathbf{b}$ OR $A^T\mathbf{y} = \mathbf{0}$ with $\mathbf{y}^T\mathbf{b} = 1$.

If $\mathbf{b}$ is not in the column space of A it is not orthogonal to the nullspace of A^T. Multiply the equations $x_1 - x_2 = 1$, $x_2 - x_3 = 1$ and $x_1 - x_3 = 1$ by numbers y_1, y_2 and y_3 chosen so that the equations add up to $0 = 1$.

Solution: Let $y_1 = 1$, $y_2 = 1$ and $y_3 = -1$. Then the left-hand side of the sum of the equations is:

$$(x_1 - x_2) + (x_2 - x_3) - (x_1 - x_3) = x_1 - x_2 + x_2 - x_3 + x_3 - x_1 = 0,$$

and the right-hand side verifies that $\mathbf{y}^T\mathbf{b} = 1$:

$$1 + 1 - 1 = 1.$$

Problem 16.2: (4.1#32.) Suppose I give you four nonzero vectors $\mathbf{r}, \mathbf{n}, \mathbf{c}$ and $\mathbf{l}$ in $\mathbb{R}^2$.

a) What are the conditions for those to be bases for the four fundamental subspaces $C(A^T)$, $N(A)$, $C(A)$, and $N(A^T)$ of a 2 by 2 matrix?

b) What is one possible matrix A?

Solution:

a) In order for $\mathbf{r}$ and $\mathbf{n}$ to be bases for $N(A)$ and $C(A^T)$, we must have

$$\mathbf{r} \cdot \mathbf{n} = 0,$$

as the row space and null space must be orthogonal. Similarly, in order for $\mathbf{c}$ and $\mathbf{l}$ to form bases for $C(A)$ and $N(A^T)$ we need

$$\mathbf{c} \cdot \mathbf{l} = 0,$$

1

as the column space and the left nullspace are orthogonal. In addition, we need:

$$\dim N(A) + \dim C(A^T) = n \quad \text{and} \quad \dim N(A^T) + \dim C(A) = m;$$

however, in this case $n = m = 1$, and as the four vectors we are given are nonzero both of these equations reduce to $1 + 1 = 2$, which is automatically satisfied.

b) One possible such matrix is $A = \mathbf{c}\mathbf{r}^T$.

Note that each column of A will be a multiple of $\mathbf{c}$, so it will have the desired column space. On the other hand, each row of A will be a multiple of $\mathbf{r}$, so A will have the desired row space. The nullspaces don't need to be checked, as any matrix with the correct row and column space will have the desired nullspaces (as the nullspaces are just the orthogonal complements of the row and column spaces).