Exercises on orthogonal matrices and Gram-Schmidt

Problem 17.1: (4.4 #10.b *Introduction to Linear Algebra:* Strang)
Orthonormal vectors are automatically linearly independent.
Matrix Proof: Show that $Q\mathbf{x} = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$. Since Q may be rectangular, you can use Q^T but not Q^{-1}.

Solution: By definition, Q is a matrix whose columns are orthonormal, and so we know that $Q^TQ = I$ (where Q may be rectangular). Then:

$$Q\mathbf{x} = \mathbf{0} \implies Q^TQ\mathbf{x} = Q^T\mathbf{0} \implies I\mathbf{x} = \mathbf{0} \implies \mathbf{x} = \mathbf{0}.$$

Thus the nullspace of Q is the zero vector, and so the columns of Q are linearly independent. There are no non-zero linear combinations of the columns that equal the zero vector. Thus, orthonormal vectors are automatically linearly independent.

Problem 17.2: (4.4 #18) Given the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ listed below, use the Gram-Schmidt process to find orthogonal vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ that span the same space.

$$\mathbf{a} = (1, -1, 0, 0), \mathbf{b} = (0, 1, -1, 0), \mathbf{c} = (0, 0, 1, -1).$$

Show that $\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$ and $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ are bases for the space of vectors perpendicular to $\mathbf{d} = (1, 1, 1, 1)$.

Solution: We apply Gram-Schmidt to $\mathbf{a}, \mathbf{b}, \mathbf{c}$. First, we set

$$\mathbf{A} = \mathbf{a} = (\mathbf{1}, \mathbf{-1}, \mathbf{0}, \mathbf{0}).$$

Next we find $\mathbf{B}$:

$$\mathbf{B} = \mathbf{b} - \frac{\mathbf{A^T b}}{\mathbf{A^T A}}\mathbf{A} = (0, 1, -1, 0) + \frac{1}{2}(1, -1, 0, 0) = \left(\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}}, \mathbf{-1}, \mathbf{0}\right).$$

And then we find $\mathbf{C}$:

$$\mathbf{C} = \mathbf{c} - \frac{\mathbf{A^T c}}{\mathbf{A^T A}}\mathbf{A} - \frac{\mathbf{B^T c}}{\mathbf{B^T B}}\mathbf{B} = (0, 0, 1, -1) + \frac{2}{3}\left(\frac{1}{2}, \frac{1}{2}, -1, 0\right) = \left(\frac{\mathbf{1}}{\mathbf{3}}, \frac{\mathbf{1}}{\mathbf{3}}, \frac{\mathbf{1}}{\mathbf{3}}, \mathbf{-1}\right).$$

We know from the first problem that the elements of the set $\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$ are linearly independent, and each vector is orthogonal to (1,1,1,1). The space of vectors perpendicular to $\mathbf{d}$ is three dimensional (since the row space of $(1, 1, 1, 1)$ is one-dimensional, and the number of dimensions of the row space added to the number of dimensions of the nullspace add to 4). Therefore $\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$ forms a basis for the space of vectors perpendicular to $\mathbf{d}$.

Similarly, $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is a basis for the space of vectors perpendicular to $\mathbf{d}$ because the vectors are linearly independent, orthogonal to (1,1,1,1), and because there are three of them.