

## Exercises on determinant formulas and cofactors

**Problem 19.1:** Compute the determinant of:

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Which method of computing the determinant do you prefer for this problem, and why?

**Solution:** The preferred method is that of using **cofactors**. We apply the Big Formula:

$$\det A = \sum_{P=(\alpha,\beta,\dots,\omega)} (\det P) a_{1\alpha} a_{2\beta} \cdots a_{n\omega}$$

to  $A$ :

$$\begin{aligned} \det A &= 0 \begin{vmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} - 0 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} + 0 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} - 1 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ &= -1 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1. \end{aligned}$$

This is quicker than row exchange:

$$\begin{aligned} \det A &= \det \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = -\det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ &= \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = -\det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= -1. \end{aligned}$$

**Problem 19.2:** (5.2 #33. *Introduction to Linear Algebra: Strang*) The symmetric Pascal matrices have determinant 1. If I subtract 1 from the  $n, n$  entry, why does the determinant become zero? (Use rule 3 or cofactors.)

$$\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} = 1 \text{ (known)} \quad \det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & \mathbf{19} \end{bmatrix} = \mathbf{0} \text{ (to explain).}$$

**Solution:** The difference in the  $n, n$  entry (in the example, the difference between 19 and 20) multiplies its cofactor, the determinant of the  $n - 1$  by  $n - 1$  symmetric Pascal matrix. In our example this matrix is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}.$$

We're told that this matrix has determinant 1. Since the  $n, n$  entry multiplies its cofactor positively, the overall determinant drops by 1 to become 0.