Exercises on eigenvalues and eigenvectors

Problem 21.1: (6.1 #19. *Introduction to Linear Algebra:* Strang) A three by three matrix B is known to have eigenvalues 0, 1 and 2. This information is enough to find three of these (give the answers where possible):

a) The rank of B

b) The determinant of $B^T B$

c) The eigenvalues of $B^T B$

d) The eigenvalues of $(B^2 + I)^{-1}$

Solution:

a) B has 0 as an eigenvalue and is therefore singular (not invertible). Since B is a three by three matrix, this means that its rank can be at most 2. Since B has two distinct nonzero eigenvalues, its rank is exactly 2.

b) Since B is singular, $\det(B) = 0$. Thus $\det(B^T B) = \det(B^T)\det(B) = 0$.

c) There is not enough information to find the eigenvalues of $B^T B$. For example:

$$\text{If } B = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 2 \end{bmatrix} \text{ then } B^T B = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 4 \end{bmatrix} \text{ and } B \text{ has eigenvalues } 0, 1, 4.$$

$$\text{If } B = \begin{bmatrix} 0 & 1 & \\ & 1 & \\ & & 2 \end{bmatrix} \text{ then } B^T B = \begin{bmatrix} 0 & & \\ & 2 & \\ & & 4 \end{bmatrix} \text{ and } B \text{ has eigenvalues } 0, 2, 4.$$

d) If $p(t)$ is a polynomial and if $\mathbf{x}$ is an eigenvector of A with eigenvalue λ, then

$$p(A)\mathbf{x} = p(\lambda)\mathbf{x}.$$

We also know that if λ is an eigenvalue of A then $1/\lambda$ is an eigenvalue of A^{-1}. Hence the eigenvalues of $(B^2 + I)^{-1}$ are $\frac{1}{0^2+1}$, $\frac{1}{1^2+1}$ and $\frac{1}{2^2+1}$, or **1, 1/2** and **1/5**.

Problem 21.2: (6.1 #29.) Find the eigenvalues of A, B, and C when

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}, \ B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}.$$

Solution: Since the eigenvalues of a triangular matrix are its diagonal entries, the eigenvalues of A are 1,4, and 6. For B we have:

$$\begin{aligned} \det(B - \lambda I) &= (-\lambda)(2-\lambda)(-\lambda) - 3(2-\lambda) \\ &= (\lambda^2 - 3)(2-\lambda). \end{aligned}$$

Hence the eigenvalues of B are $\pm\sqrt{3}$ and 2. Finally, for C we have:

$$\begin{aligned} \det(C - \lambda I) &= (2-\lambda)[(2-\lambda)^2 - 4] - 2[2(2-\lambda) - 4] + 2[4 - 2(2-\lambda)] \\ &= \lambda^3 - 6\lambda^2 = \lambda^2(\lambda - 6). \end{aligned}$$

The eigenvalues of C are 6, 0, and 0.

We can quickly check our answers by computing the determinants of A and B and by noting that C is singular.