Exercises on diagonalization and powers of A

Problem 22.1: (6.2 #6. *Introduction to Linear Algebra:* Strang) Describe all matrices S that diagonalize this matrix A (find all eigenvectors):

$$A = \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix}.$$

Then describe all matrices that diagonalize A^{-1}.

Solution: To find the eigenvectors of A, we first find the eigenvalues:

$$\det \begin{bmatrix} 4-\lambda & 0 \\ 1 & 2-\lambda \end{bmatrix} = 0 \Longrightarrow (4-\lambda)(2-\lambda) = 0.$$

Hence the eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = 2$. Using these values, we find the eigenvectors by solving $(A - \lambda I)\mathbf{x} = \mathbf{0}$:

$$(A - \lambda_1 I)\mathbf{x} = \begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow y = 2z,$$

thus any multiple of **(2,1)** is an eigenvector for λ_1.

$$(A - \lambda_2 I)\mathbf{x} = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow y = 0, z = \text{free variable},$$

thus any multiple of **(0,1)** is an eigenvector for λ_2. Therefore the columns of the matrices S that diagonalize A are nonzero multiples of (2,1) and (1,0). They can appear in either order.

Finally, because $A^{-1} = S\Lambda^{-1}S^{-1}$ the same matrices S will diagonalize A^{-1}.

Problem 22.2: (6.2 #16.) Find Λ and S to diagonalize A :

$$A = \begin{bmatrix} .6 & .9 \\ .4 & .1 \end{bmatrix}.$$

What is the limit of Λ^k as $k \to \infty$? What is the limit matrix of $S\Lambda^k S^{-1}$? In the columns of this matrix you see the ________.

Solution: Since each of the columns of A sum to one, A is a Markov matrix and definitely has eigenvalue $\lambda_1 = 1$. The trace of A is .7, so the other eigenvalue is $\lambda_2 = .7 - 1 = -.3$. To find S we need to find the corresponding eigenvectors:

$$(A - \lambda_1 I)\mathbf{x_1} = \begin{bmatrix} -.4 & .9 \\ .4 & -.9 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow \mathbf{x_1} = (9, 4).$$

$$(A - \lambda_2 I)\mathbf{x_2} = \begin{bmatrix} .9 & .9 \\ .4 & .4 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow y = -z \Longrightarrow \mathbf{x_2} = (1, -1).$$

Putting these together, we have:

$$S = \begin{bmatrix} 9 & 1 \\ 4 & -1 \end{bmatrix} \text{ and } \Lambda = \begin{bmatrix} 1 & \\ & -.3 \end{bmatrix}. \text{ As } k \to \infty, \ \ \Lambda^k \to \begin{bmatrix} 1 & \\ & 0 \end{bmatrix}.$$

So

$$S\Lambda^k S^{-1} \to \begin{bmatrix} 9 & 1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & \\ & 0 \end{bmatrix} \left(\frac{1}{13}\right) \begin{bmatrix} 9 & 1 \\ 4 & -9 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} 9 & 9 \\ 4 & 4 \end{bmatrix}.$$

In the columns of this matrix you see the **steady state vector.**