Exercises on differential equations and e^{At}

Problem 23.1: (6.3 #14.a *Introduction to Linear Algebra:* Strang) The matrix in this question is skew-symmetric $(A^T = -A)$:

$$\frac{d\mathbf{u}}{dt} = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} \mathbf{u} \quad \text{or} \quad \begin{array}{l} {u_1}' = cu_2 - bu_3 \\ {u_2}' = au_3 - cu_1 \\ {u_3}' = bu_1 - au_2. \end{array}$$

Find the derivative of $||\mathbf{u}(t)||^2$ using the definition:

$$||\mathbf{u}(t)||^2 = {u_1}^2 + {u_2}^2 + {u_3}^2.$$

What does this tell you about the rate of change of the length of $\mathbf{u}$? What does this tell you about the range of values of $\mathbf{u}(t)$?

Solution:

$$\begin{aligned} \frac{d||\mathbf{u}(t)||^2}{dt} &= \frac{d({u_1}^2 + {u_2}^2 + {u_3}^2)}{dt} \\ &= 2u_1{u_1}' + 2u_2{u_2}' + 2u_3{u_3}' \\ &= 2u_1(cu_2 - bu_3) + 2u_2(au_3 - cu_1) + 2u_3(bu_1 - au_2) \\ &= 0. \end{aligned}$$

This means $||\mathbf{u}(t)||^2$ stays equal to $||\mathbf{u}(0)||^2$. Because $\mathbf{u}(t)$ never changes length, it is always on the circumference of a circle of radius $||\mathbf{u}(0)||$.

Problem 23.2: (6.3 #24.) Write $A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$ as $S\Lambda S^{-1}$. Multiply $Se^{\Lambda t}S^{-1}$ to find the matrix exponential e^{At}. Check your work by evaluating e^{At} and the derivative of e^{At} when $t = 0$.

Solution: The eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 3$, with corresponding eigenvectors $\mathbf{x_1} = (1, 0)$ and $\mathbf{x_2} = (1, 2)$. This gives us the following values for S, Λ, and S^{-1} :

$$S = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, S^{-1} = \begin{bmatrix} 1 & -1/2 \\ 0 & 1/2 \end{bmatrix}.$$

We use these to find e^{At} :

$$Se^{\Lambda t}S^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1 & -1/2 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} e^t & .5e^{3t} - .5e^t \\ 0 & e^{3t} \end{bmatrix} = e^{At}.$$

Check:

$$e^{At} = \begin{bmatrix} e^t & .5e^{3t} - .5e^t \\ 0 & e^{3t} \end{bmatrix} \text{ equals } I \text{ when } t = 0. \checkmark$$

$$\frac{de^{At}}{dt} = \begin{bmatrix} e^t & 1.5e^{3t} - .5e^t \\ 0 & 3e^{3t} \end{bmatrix}.$$

$$\left.\frac{de^{At}}{dt}\right|_{t=0} = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} = A. \checkmark$$