Exercises on Markov matrices; Fourier series

Problem 24.1: (6.4 #7. *Introduction to Linear Algebra:* Strang)

a) Find a symmetric matrix $\begin{bmatrix} 1 & b \\ b & 1 \end{bmatrix}$ that has a negative eigenvalue.

b) How do you know it must have a negative pivot?

c) How do you know it can't have two negative eigenvalues?

Solution:

a) The eigenvalues of that matrix are $1 \pm b$. If $b > 1$ or $b < -1$ the matrix has a negative eigenvalue.

b) The pivots have the same signs as the eigenvalues. If the matrix has a negative eigenvalue, then it must have a negative pivot.

c) To obtain one negative eigenvalue, we choose either $b > 1$ or $b < -1$ (as stated in part (a)). If we choose $b > 1$, then $\lambda_1 = 1 + b$ will be positive while $\lambda_2 = 1 - b$ will be negative. Alternatively, if we choose $b < -1$, then $\lambda_1 = 1 + b$ will be negative while $\lambda_2 = 1 - b$ will be positive. Therefore this matrix cannot have two negative eigenvalues.

Problem 24.2: (6.4 #23.) Which of these classes of matrices do A and B belong to: invertible, orthogonal, projection, permutation, diagonalizable, Markov?

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad B = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Which of these factorizations are possible for A and B: LU, QR, $S\Lambda S^{-1}$, or $Q\Lambda Q^T$?

Solution:

a) For A :

$\det A = -1 \neq 0$.	A is **invertible**.
$AA^T = I$.	A is **orthogonal**.
$A^2 = I \neq A$.	A is **not a projection**.
A has one 1 in each row and column with 0's elsewhere.	A is **a permutation**.
$A = A^T$, so A is symmetric.	A is **diagonalizable**.
Each column of A sums to one.	A is **Markov**.

All of the factorizations are possible for A: LU and QR are always possible, $S\Lambda S^{-1}$ is possible because it is diagonalizable, and $Q\Lambda Q^T$ is possible because it is symmetric.

b) For B :

$\det B = 0$.	B is **not invertible**.
$BB^T \neq I$.	B is **not orthogonal**.
$B^2 = B$.	B is **a projection**.
B does not have one 1 in each row and each column, with 0's elsewhere.	B is **not a permutation**.
$B = B^T$ so B is symmetric.	B is **diagonalizable**.
Each column of B sums to one.	B is **Markov**.

All of the factorizations are possible for B: LU and QR are always possible, $S\Lambda S^{-1}$ is possible because it is diagonalizable, and $Q\Lambda Q^T$ is possible because it is symmetric.

Problem 24.3: (8.3 #11.) Complete A to a Markov matrix and find the steady state eigenvector. When A is a symmetric Markov matrix, why is $\mathbf{x}_1 = (1, \ldots, 1)$ its steady state?

$$A = \begin{bmatrix} .7 & .1 & .2 \\ .1 & .6 & .3 \\ \text{---} & \text{---} & \text{---} \end{bmatrix}.$$

Solution: Matrix A becomes:

$$A = \begin{bmatrix} .7 & .1 & .2 \\ .1 & .6 & .3 \\ \mathbf{.2} & \mathbf{.3} & \mathbf{.5} \end{bmatrix},$$

with steady state vector (1,1,1). When A is a *symmetric* Markov matrix, the elements of each row sum to one. The elements of each row of $A - I$ then sum to zero. Since the steady state vector $\mathbf{x}$ is the eigenvector associated with eigenvalue $\lambda = 1$, we solve $(A - \lambda I)\mathbf{x} = (A - I)\mathbf{x} = \mathbf{0}$ to get $\mathbf{x} = (1, \ldots, 1)$.