Exercises on symmetric matrices and positive definiteness

Problem 25.1: (6.4 #10. *Introduction to Linear Algebra:* Strang) Here is a quick "proof" that the eigenvalues of all real matrices are real:

False Proof: $A\mathbf{x} = \lambda\mathbf{x}$ gives $\mathbf{x}^T A\mathbf{x} = \lambda\mathbf{x}^T\mathbf{x}$ so $\lambda = \dfrac{\mathbf{x}^T A\mathbf{x}}{\mathbf{x}^T\mathbf{x}}$ is real.

There is a hidden assumption in this proof which is not justified. Find the flaw by testing each step on the 90° rotation matrix:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

with $\lambda = i$ and $\mathbf{x} = (i, 1)$.

Solution: We can esily confirm that $A\mathbf{x} = \lambda\mathbf{x} = \begin{bmatrix} -1 \\ i \end{bmatrix}$. Next, check if $\mathbf{x}^\mathbf{T} A\mathbf{x} = \lambda\mathbf{x}^\mathbf{T}\mathbf{x}$ is true for the 90° rotation matrix:

$$\begin{aligned} \mathbf{x}^T A\mathbf{x} &= \begin{bmatrix} i & 1 \end{bmatrix} \begin{bmatrix} -1 \\ i \end{bmatrix} = 0 \\ \lambda\mathbf{x}^T\mathbf{x} &= i \begin{bmatrix} i & 1 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = 0 \\ \mathbf{x}^T A\mathbf{x} &= \lambda\mathbf{x}^T\mathbf{x}. \checkmark \end{aligned}$$

Note that $\mathbf{x}^\mathbf{T}\mathbf{x} = 0$. Since the next and last step involves dividing by this term, the hidden assumption must be that $\mathbf{x}^\mathbf{T}\mathbf{x} \neq 0$. If $x = (a, b)$ then

$$\mathbf{x}^\mathbf{T}\mathbf{x} = \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = a^2 + b^2.$$

The "proof" assumes that the squares of the components of the eigenvector cannot sum to zero: $a^2 + b^2 \neq 0$. This may be false if the components are complex.

Problem 25.2: (6.5 #32.) A *group* of nonsingular matrices includes AB and A^{-1} if it includes A and B. "Products and inverses stay in the group." Which of these are groups?

a) Positive definite symmetric matrices A.

b) Orthogonal matrices Q.

c) All exponentials e^{tA} of a fixed matrix A.

d) Matrices D with determinant 1.

Solution:

a) The positive definite symmetric matrices A **do not form a group.** To show this, we provide a counterexample in the form of two positive definite symmetric matrices A and B whose product is not a positive definite symmetric matrix.

If $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$ then $AB = \begin{bmatrix} 2.5 & 2 \\ 1.5 & 1.5 \end{bmatrix}$ is not symmetric.

b) The orthogonal matrices Q **form a group**. If A and B are orthogonal matrices, then:

$$A^T A = I \Rightarrow A^{-1} = A^T \Rightarrow A^{-1} \text{ is orthogonal, and}$$
$$B^T B = I \Rightarrow (AB)^T AB = B^T A^T AB = B^T B = I \Rightarrow AB \text{ is orthogonal.}$$

c) The exponentials e^{tA} of a fixed matrix A **form a group.** For the elements e^{pA} and e^{qA}:

$$(e^{pA})^{-1} = e^{-pA} \text{ is of the form } e^{tA}$$
$$e^{pA}e^{qA} = e^{(p+q)A} \text{ is of the form } e^{tA}$$

d) The matrices D with determinant 1 **form a group.** If $\det A = 1$ then $\det A^{-1} = 1$. If matrices A and B have determinant 1 then their product also has determinant 1:

$$\det(AB) = \det(A)\det(B) = 1.$$