

Exercises on positive definite matrices and minima

Problem 27.1: (6.5 #33. *Introduction to Linear Algebra: Strang*) When A and B are symmetric positive definite, AB might not even be symmetric, but its eigenvalues are still positive. Start from $AB\mathbf{x} = \lambda\mathbf{x}$ and take dot products with $B\mathbf{x}$. Then prove $\lambda > 0$.

Solution:

$$\begin{aligned}AB\mathbf{x} &= \lambda\mathbf{x} \\(AB\mathbf{x})^T B\mathbf{x} &= (\lambda\mathbf{x})^T B\mathbf{x} \\(B\mathbf{x})^T A^T B\mathbf{x} &= \lambda\mathbf{x}^T B\mathbf{x} \\(B\mathbf{x})^T A(B\mathbf{x}) &= \lambda(\mathbf{x}^T B\mathbf{x}).\end{aligned}$$

where $A^T = A$ because A is symmetric. Since A is positive definite we know $(B\mathbf{x})^T A(B\mathbf{x}) > 0$, and since B is positive definite $\mathbf{x}^T B\mathbf{x} > 0$. Hence, λ must be positive as well.

Problem 27.2: Find the quadratic form associated with the matrix $\begin{bmatrix} 1 & 5 \\ 7 & 9 \end{bmatrix}$.

Is this function $f(x, y)$ always positive, always negative, or sometimes positive and sometimes negative?

Solution: To find the quadratic form, compute $\mathbf{x}^T A\mathbf{x}$:

$$\begin{aligned}f(x, y) &= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 7 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\&= x(x + 5y) + y(7x + 9y) \\&= \mathbf{x}^2 + \mathbf{12xy} + \mathbf{9y}^2.\end{aligned}$$

This expression can be positive, e.g. when $y = 0$ and $x \neq 0$.

The expression will sometimes be negative because A is not positive definite. For instance, $f(2, -2) = -8$. Thus the quadratic form associated with the matrix A is **sometimes positive and sometimes negative**. Another way to reach this conclusion is to note that $\det A = -26$ is negative and so A is not positive definite.