

Exercises on similar matrices and Jordan form

Problem 1.1: (6.6 #12. *Introduction to Linear Algebra*: Strang) These Jordan matrices have eigenvalues 0, 0, 0, 0. They have two eigenvectors; one from each block. However, their block sizes don't match and they are *not similar*:

$$J = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] \text{ and } K = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array} \right].$$

For a generic matrix M , show that if $JM = MK$ then M is not invertible and so J is not similar to K .

Solution: Let $M = (m_{ij})$. Then:

$$JM = \begin{bmatrix} m_{21} & m_{22} & m_{23} & m_{24} \\ 0 & 0 & 0 & 0 \\ m_{41} & m_{42} & m_{43} & m_{44} \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } MK = \begin{bmatrix} 0 & m_{11} & m_{12} & 0 \\ 0 & m_{21} & m_{22} & 0 \\ 0 & m_{31} & m_{32} & 0 \\ 0 & m_{41} & m_{42} & 0 \end{bmatrix}$$

If $JM = MK$, then $m_{11} = m_{22} = 0$, $m_{21} = 0$, $m_{31} = m_{42} = 0$, and $m_{41} = 0$. Thus, the first column of M is all zeros and M is not invertible.

If J were similar to K there would be an invertible matrix M that satisfies $K = M^{-1}JM$, and so $MK = JM$. We just showed that there can be no such invertible matrix M . Therefore J is not similar to K .

Problem 1.2: (6.6 #20.) Why are these statements all true?

- a) If A is similar to B then A^2 is similar to B^2 .
- b) A^2 and B^2 can be similar when A and B are not similar (try $\lambda = 0, 0$.)
- c) $\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ is similar to $\begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}$.
- d) $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ is not similar to $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$.

- e) Given a matrix A , let B be the matrix obtained by exchanging rows 1 and 2 of A and then exchanging columns 1 and 2 of A . Show that A is similar to B .

Solution:

- a) If A is similar to B , then:

$$A = M^{-1}BM \implies A^2 = M^{-1}BM(M^{-1}BM) = M^{-1}B^2M.$$

Since $A^2 = M^{-1}B^2M$, by definition A^2 is similar to B^2 .

- b) Let:

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then $A^2 = B^2$ and so A^2 is similar to B^2 , but A is not similar to B because nothing but the zero matrix is similar to the zero matrix.

- c) There are multiple ways to verify that the given matrices are similar. One way is to explicitly find the matrix M that satisfies the similarity condition:

$$\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

- d) The given matrices are not similar because for the first matrix every vector in the plane is an eigenvector for $\lambda = 3$, whereas the second matrix only has a line ($y = 0$) of eigenvectors corresponding to $\lambda = 3$. We know that similar matrices have the same number of independent eigenvectors, so the given matrices cannot be similar.

- e) To exchange the first two rows of A , we multiply A on the left by:

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

In order to exchange the first two columns of A , we multiply it on the right by the same matrix M . We thus have $B = MAM$. Since $M^{-1} = M$, B is similar to A .