Exercises on singular value decomposition

Problem 29.1: (Based on 6.7 #4. *Introduction to Linear Algebra:* Strang) Verify that if we compute the singular value decomposition $A = U\Sigma V^T$ of the Fibonacci matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$,

$$\Sigma = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{\sqrt{5}-1}{2} \end{bmatrix}.$$

Solution:

$$A^T A = AA^T = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

The eigenvalues of this matrix are the roots of $x^2 - 3x + 1$, which are $\dfrac{3 \pm \sqrt{5}}{2}$. Thus we have:

$$\sigma_1^2 = \frac{3+\sqrt{5}}{2} \quad \text{and} \quad \sigma_2^2 = \frac{3-\sqrt{5}}{2}.$$

To check that $\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$, we will square the entries of the matrix Σ given above.

$$\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1+2\sqrt{5}+5}{4} = \frac{3+\sqrt{5}}{2}. \checkmark$$

$$\left(\frac{\sqrt{5}-1}{2}\right)^2 = \frac{5-2\sqrt{5}+1}{4} = \frac{3-\sqrt{5}}{2}. \checkmark$$

Problem 29.2: (6.7 #11.) Suppose A has orthogonal columns $\mathbf{w}_1$, $\mathbf{w}_2$, ..., $\mathbf{w}_n$ of lengths σ_1, σ_2, ..., σ_n. Calculate $A^T A$. What are U, Σ, and V in the SVD?

Solution: Since the columns of A are orthogonal, $A^T A$ is a diagonal matrix with entries ${\sigma_1}^2$, ..., ${\sigma_n}^2$. Since $A^T A = V\Sigma^2 V^T$, we find that Σ^2 is the matrix with diagonal entries ${\sigma_1}^2$, ...,${\sigma_n}^2$ and thus that Σ is the matrix with diagonal entries σ_1, ..., σ_n.

Referring again to the equation $A^T A = V\Sigma^2 V^T$, we conclude also that $V = I$.

The equation $A = U\Sigma V^T$ then tells us that U must be the matrix whose columns are $\frac{1}{\sigma_i}\mathbf{w}_i$.