Exercises on left and right inverses; pseudoinverse

Problem 26.1: Find a right inverse for $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Solution: We apply the formula $A_{\text{right}}^{-1} = A^T(AA^T)^{-1}$:

$$\begin{aligned} A^T &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \\ AA^T &= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \\ (AA^T)^{-1} &= \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \\ A^T(AA^T)^{-1} &= \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \\ 1/2 & 0 \end{bmatrix}. \end{aligned}$$

Thus, $A_{\text{right}}^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \\ 1/2 & 0 \end{bmatrix}$ is one right inverse of A. We can quickly check that $AA_{\text{right}}^{-1} = I$.

Problem 26.2: Does the matrix $A = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}$ have a left inverse? A right inverse? A pseudoinverse? If the answer to any of these questions is "yes", find the appropriate inverse.

Solution: The second row of A is a multiple of the first row, so A has rank 1 and $\det A = 0$. Because A is a square matrix its determinant is defined, and we can use the fact that $\det C \cdot \det D = \det(CD)$ to prove that A can't have a left or right inverse. (If $AB = I$, then $\det A \det B = \det I$ implies $0 = 1$.)

We *can* find a pseudoinverse $A^+ = V\Sigma^+U^T$ for A. We start by finding the singular value decomposition $U\Sigma V^T$ of A.

The SVD of A was calculated in the lecture on singular value decomposition, so we know that

$$\underset{A}{\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}} = \underset{U}{\frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}} \underset{\Sigma}{\begin{bmatrix} \sqrt{125} & 0 \\ 0 & 0 \end{bmatrix}} \underset{V^T}{\begin{bmatrix} .8 & .6 \\ .6 & -.8 \end{bmatrix}}.$$

Hence, $\Sigma^+ = \begin{bmatrix} 1/\sqrt{125} & 0 \\ 0 & 0 \end{bmatrix}$ and

$$\begin{aligned} A^+ &= V\Sigma^+U^T \\ &= \begin{bmatrix} .8 & .6 \\ .6 & -.8 \end{bmatrix} \begin{bmatrix} 1/\sqrt{125} & 0 \\ 0 & 0 \end{bmatrix} \left(\frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \right) \\ &= \begin{bmatrix} .8 & .6 \\ .6 & -.8 \end{bmatrix} \begin{bmatrix} 1/25 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \\ &= \begin{bmatrix} .8 & .6 \\ .6 & -.8 \end{bmatrix} \begin{bmatrix} 1/25 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \\ &= \frac{1}{125} \begin{bmatrix} 4 & 8 \\ 3 & 6 \end{bmatrix}. \end{aligned}$$

To check our work, we confirm that A^+ reverses the operation of A on its row space using the bases we found while computing its SVD. Recall that

$$A\mathbf{v}_j = \begin{cases} \sigma_j \mathbf{u}_j & \text{for } j \le r \\ \mathbf{0} & \text{for } j > r. \end{cases}$$

Here $\mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$ and $A^+\mathbf{u}_1 = \frac{1}{\sqrt{125}} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \frac{1}{\sigma_1}\mathbf{v}_1$. We can also check that $A^+\mathbf{u}_2 = \frac{1}{125} \begin{bmatrix} 4 & 8 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \mathbf{0}$.