

### Exercises on independence, basis, and dimension

**Problem 9.1:** (3.5 #2. *Introduction to Linear Algebra: Strang*) Find the largest possible number of independent vectors among:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix},$$
$$\mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \text{ and } \mathbf{v}_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

**Solution:** Since  $\mathbf{v}_4 = \mathbf{v}_2 - \mathbf{v}_1$ ,  $\mathbf{v}_5 = \mathbf{v}_3 - \mathbf{v}_1$ , and  $\mathbf{v}_6 = \mathbf{v}_3 - \mathbf{v}_2$ , the vectors  $\mathbf{v}_4$ ,  $\mathbf{v}_5$ , and  $\mathbf{v}_6$  are dependent on the vectors  $\mathbf{v}_1$ ,  $\mathbf{v}_2$  and  $\mathbf{v}_3$ . To determine the relationship between the vectors  $\mathbf{v}_1$ ,  $\mathbf{v}_2$  and  $\mathbf{v}_3$  we apply row reduction to the matrix  $[\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3]$ :

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

As there are three pivots, the vectors  $\mathbf{v}_1$ ,  $\mathbf{v}_2$ , and  $\mathbf{v}_3$  are independent. Therefore the largest number of independent vectors among the given six vectors is **three**.

**Problem 9.2:** (3.5 #20.) Find a basis for the plane  $x - 2y + 3z = 0$  in  $\mathbb{R}^3$ . Then find a basis for the intersection of that plane with the  $xy$  plane. Then find a basis for all vectors perpendicular to the plane.

**Solution:** This plane is the nullspace of the matrix

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The special solutions to  $A\mathbf{x} = \mathbf{0}$  are

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

These form a basis for the nullspace of  $A$  and thus for the plane.

The intersection of this plane with the  $xy$  plane contains  $\mathbf{v}_1$  and does not contain  $\mathbf{v}_2$ ; the intersection must be a line. Since  $\mathbf{v}_1$  lies on this line it also provides a basis for it.

Finally, we can use “inspection” or the cross product to find the vector

$$\mathbf{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix},$$

which is perpendicular to both  $\mathbf{v}_1$  and  $\mathbf{v}_2$ . It is therefore perpendicular to the plane. Since the space of vectors perpendicular to a plane in  $\mathbb{R}^3$  is one-dimensional,  $\mathbf{v}_3$  serves as a basis for that space.