Self-Paced Study Guide in Algebra

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1 How to Use This Guide

The *Self-Paced Review* consists of review modules with exercises; problems and solutions; self-tests and solutions; and self-evaluations for the four topic areas Algebra, Geometry and Analytic Geometry, Trigonometry, and Exponentials & Logarithms. In addition, previous *Diagnostic Exams* with solutions are included. Each topic area is independent of the others.

The *Review Modules* are designed to introduce the core material for each topic area. A numbering system facilitates easy tracking of subject material. For example, in Algebra, the subtopic Linear Equations is numbered with 2.3. Problems and the self-evaluations are categorized using this numbering system.

When using the *Self-Paced Review*, it is important to differentiate between concept learning and problem solving. The review modules are oriented toward refreshing concept understanding while the problems and self-tests are designed to develop problems solving ability. When reviewing the modules, exercises are liberally sprinkled throughout the modules which should be solved while working through the module. The problems should be attempted without looking at the solutions. If a problem cannot be solved after at least two honest efforts, then consult the solutions. The tests should be taken only when both an understanding of the material and a problem solving ability have been achieved. The self-evaluation is a useful tool to evaluate one's mastery of the material. The previous Diagnostic Exams should provide the finishing touch.

The review modules were written by Professor A. P. French (Physics Department) and Adeliada Moranescu (MIT Class of 1994). The problems and solutions were written by Professor Arthur Mattuck (Mathematics Department). This document was originally produced by the Undergraduate Academic Affairs Office, August, 1992, and edited and transcribed to LaTeX by Tea Dorminy (MIT Class of 2013) in August, 2010.

2 Algebra Self-Paced Review Module

One widely used algebra textbook[1] begins with the sentence: "Algebra is really arithmetic in disguise." Arithmetic makes use of specific numbers; algebra develops general results that can be applied regardless of what the particular numbers are. That makes algebra much more powerful than arithmetic. But if you want to *apply* a given algebraic result, you have to replace the xs and ys by specific numbers, and you have to know how to handle numerical quantities. So this algebra review begins with some stuff about numbers as such.

2.1 Scientific Notation

Writing numbers in what is called "scientific notation" is an absolute necessity in science and engineering. You will be using it all the time. It is based on the fact that any positive number, however large or however small, can be written as a number between 1 and 10 multiplied by a power of 10. (And, for negative numbers, we simply put a $-$ sign in front.)

Begin by recalling that:

$$\begin{aligned}
10^1 &= 10 \\
10^2 &= 10 \cdot 10 = 100 \\
10^3 &= 10 \cdot 10 \cdot 10 = 1000 \\
&\text{etc.}
\end{aligned}$$

To *multiply* together any two powers of 10, we simply *add* the exponents:

$$(10^m)(10^n) = 10^{(m+n)}$$

[These and related matters are discussed in more detail in the module on Exponentials & Logarithms.]

To *divide* one power of 10 by another, we *subtract* the exponents:

$$\frac{10^m}{10^n} = 10^{m-n}.$$

Thus $10^9/10^4 = 10^{(9-4)} = 10^5 = 100000$.

[1] Hughes-Hallett, *The Math Workshop: Algebra*, W. W. Norton Co., New York, 1980.

It is implicit in this that the reciprocal of a positive power of 10 is an equal negative power:

$$\frac{1}{10^n} = 10^{-n}.$$

The powers-of-10 notation also defines what is meant by 10^0:

$$10^0 = \frac{10^n}{10^n} = 1.$$

This is all very simple, but you need to be careful when negative powers of 10 are involved. Give yourself some practice by doing the following exercises:

Exercise 2.1.1:
Write the following numbers in scientific notation:

a) 80,516

b) 0.0751

c) 3,520,000

d) 0.000 000 081.

Note: The answers to the exercises are all collected together at the end of this module. We have tried to eliminate errors, but if you find anything that you think needs to be corrected, please write to us.

A Note about Notation: We are using the center dot, e.g., $1.5 \cdot 10^4$, when we write a number as the product of a decimal with a power of 10. Perhaps you are accustomed to using the multiplication sign, $\times$, for this purpose, and you will find that both conventions are widely used. We prefer to use the center dot (especially in algebra and, later, in calculus) or, in some places, a parenthesis, to avoid any possible confusion with the variable x.

Exercise 2.1.2:
Evaluate:

a) $10^9 \cdot 10^{-3}$

b) $10^7/10^{-4}$

c) $10^{-19}/10^{-34}$

Answers? You don't need *us* to provide them! Just get the practice.

Now consider *multiplying or dividing* two numbers that are *not* pure powers of 10.

Take, for example, $2.6 \cdot 10^3$ and $5.3 \cdot 10^4$.

Their *product* is given by:

$$(2.6 \cdot 10^3)(5.3 \cdot 10^4) = (2.6)(5.3)(10^3 \cdot 10^4) = 13.8 \cdot 10^{3+4} = 13.8 \cdot 10^7 = 1.38 \times 10^8.$$

Their *quotient* is given by:

$$\frac{2.6 \cdot 10^3}{5.3 \cdot 10^4} = \left(\frac{2.6}{5.3}\right) 10^{3-4} = 0.68 \cdot 10^{-1} = 6.8 \cdot 10^{-2}.$$

Notice how we don't stop until we have converted the answer into a number between 1 and 10 multiplied by a power of 10.

Exercise 2.1.3:
Evaluate, in scientific notation:

a) $(1.5 \cdot 10^4)(7.5 \cdot 10^{-5})$

b) $(4.3 \cdot 10^{-6})/(3.1 \cdot 10^{-10})$

c) $\frac{(1.2 \cdot 10^{-5})(1.5 \cdot 10^3)}{9 \times 10^{-9}}$

To *add* or *subtract* numbers in scientific notation, you have first to re-arrange the numbers so that all the powers of 10 are the same; then you can add or subtract the decimal parts, leaving the power of 10 alone. It's usually best if you first express all the numbers in terms of the *highest* power of 10. (In this connection, remember that, with negative powers of 10, smaller exponents means bigger numbers: 10^{-3} is bigger than 10^{-5}.) You may, however, need to make a final adjustment if the combination of the decimal numbers is more than 10 or less than 1.

Examples:

$$2.1 \cdot 10^3 + 3.5 \cdot 10^5 = (0.021 + 3.5) \cdot 10^5 = 3.521 \cdot 10^5$$
$$1.3 \cdot 10^8 - 8.4 \cdot 10^7 = (1.3 - 0.84) \cdot 10^8 = 0.46 \cdot 10^8 = 4.6 \cdot 10^7$$
$$9.5 \cdot 10^{-11} + 9.8 \cdot 10^{-12} = (9.5 + 0.98) \cdot 10^{-11} = 10.48 \cdot 10^{-11} = 1.048 \cdot 10^{-10}$$

Exercise 2.1.4:
Evaluate:

a) $9.76 \cdot 10^9 + 7.5 \cdot 10^8$

b) $1.25 \cdot 10^6 - 7.85 \cdot 10^5$

c) $4.21 \cdot 10^{25} - 1.85 \cdot 10^{26}$

d) $4.05 \cdot 10^{-19} - 10^{-20}$

e) $(1.2 \cdot 10^{-19})(5.2 \cdot 10^{10} + 4 \cdot 10^9)$

2.2 Significant Figures

In the above examples and exercises in adding or subtracting numbers expressed in scientific notation, you will have noticed that you may end up with a large number of digits. But not all of these may be significant. For example, when we added $2.1 \cdot 10^3$ and $3.5 \cdot 10^5$, we got $3.521 \cdot 10^5$. However, each of the numbers being combined were given with only two-digit accuracy. (We are assuming that 3.5 means simply that the number is closer to 3.5 than it is to 3.4 or 3.6. If it meant 3.500 then these extra zero digits should have been included.) This means that the $2.1 \cdot 10^3$ did not add anything significant to the bigger number, and we were not justified in giving more than two digits in the final answer, which therefore should have been given as just $3.5 \cdot 10^5$.

Note, however, that if we were asked to add, say, $8.6 \cdot 10^3$ to $3.5 \cdot 10^5$, the smaller number would make a significant contribution to the final answer. The straight addition would give us $3.586 \cdot 10^5$. Rounding this off to two digits would then give the answer $3.6 \cdot 10^5$.

The general rules governing significant figures are:

1. The final answer should not contain more digits than are justified by the least accurate of the numbers being combined; **but**

2. Accuracy contained in the numbers being combined should not be sacrificed in the rounding-off process.

A few examples will help spell out these conditions. (We'll ignore the powers-of-10 factors for this purpose):

Addition:	$1.63 + 2.1789 + 0.96432 = 4.77422$, rounded to 4.77.
Subtraction:	$113.2 - 1.43 = 111.77$, rounded to 111.8.
Multiplication:	$(11.3)(0.43) = 4.859$, rounded to 4.9 (only two digits justified). BUT $(11.3)(0.99) = 11.187$, rounded to 11.2 (three digits — because rounding to two digits would imply only 10% accuracy, whereas each of the numbers being multiplied would justify 1%)
Division:	$\frac{1.30}{0.43} = 3.02325814$, rounded to 3.0 (two digits). BUT $(1.30)/(0.99) = 1.14141414141$, rounded to 1.14 (three digits justified).

Beware of your calculator! In the last two examples we used a pocket calculator to do the divisions. This automatically gave 10 digits in each answer. *Most of these digits are **in**significant*! Whenever you use your calculator for such a purpose, always ask yourself how many digits are justified and should be retained in the answer. Most of the calculations you will be doing will probably involve numbers with only a few significant digits. Get into the habit of cutting down your final answers to the proper size. As you can see, this situation will arise most importantly when you are doing divisions with your calculator. But watch out for surplus digits in multiplications as well.

Now let's turn to algebra proper.

2.3 Linear Equations

2.3.1 Equations in one variable

These scarcely need any discussion. There is one unknown, say x, and an equation that relates this to given numbers or constants. The only job is to tidy things up so as to solve for x explicitly.

Exercise 2.3.1:
Solve for x:

a) $5\left(x + \frac{1}{4}\right) = 2x - \frac{1}{8}$

b) $\frac{3}{x} - \frac{4}{5} = \frac{1}{x} + \frac{1}{3}$

c) $3(ax + b) = 5bx + c$

2.3.2 Simultaneous Equations in Two Variables

However these equations are originally written, they can always be reduced to the following form:

$$a_1 x + b_1 y = c_1$$
$$a_2 x + b_2 y = c_2$$

where the coefficients $a_1, a_2, b_1, b_2, c_1, c_2$ may be positive or negative.

Two essentially equivalent methods can be used to obtain the solutions for x and y:

1. *Substitution:* Use one of the equations (say the first) to get one of the unknowns (say y) in terms of x and known quantities. Substitute into the other equation to solve for the remaining unknown (x). Plug this value of x back into either of the initial equations to get y.

2. *Elimination:* Multiply the original equations by factors that make the coefficient of one unknown (say y) the same in both. By subtraction, eliminate y; this leads at once to x:

Multiply 1st eq. by b_2:	$a_1b_2x + b_1b_2y$	$=$	b_2c_1
Multiply 2nd eq. by b_1:	$a_2b_1x + b_2b_1y$	$=$	b_1c_2
Subtract:	$(a_1b_2 - a_2b_1)x$	$=$	$b_2c_1 - b_1c_2$ Then solve
Thus:	x	$=$	$\frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}$.

for y as before. But the last equation above points to a special situation: If the denominator $(a_1b_2 - a_2b_1) = 0$, then x becomes indeterminate, and therefore so does y.

[**Keep it neat!** Good housekeeping in mathematics is very important. It will make it easier for you to check your work and it will help you to avoid errors. Notice how we put the equals signs underneath one another in the above analysis. Make a practice of doing this yourself. Concentrate on making this a habit, and don't throw it out the window when you are taking a quiz or exam. You're bound to benefit from being neat and orderly.]

Exercise 2.3.2:
Solve for both unknowns:

[t]

a) $$\begin{aligned} 2x - 3y &= 4 \\ 3x - 2y &= 5 \end{aligned}$$

b) $$\begin{aligned} 5a &= b - 6 \\ 2b &= a + 4 \end{aligned}$$

c) $$\begin{aligned} \frac{3}{x} + \frac{4}{y} &= 5 \\ \frac{4}{x} - \frac{2}{y} &= 3 \end{aligned}$$

[In (c), resist the temptation to multiply both equations throughout by xy to clear the denominators. Just put $1/x = u$, $1/y = v$, and solve first for u and v, which are just as legitimate variables as x and y.]

2.4 Polynomials

Much of what you do in algebra (and later, in calculus) will be based on a familiarity with expressions made up of a sum of terms like $10x^3$ or $(3.2)y^5$ — in other words, sums of products of numbers called coefficients and powers of variables such as x. Such an expression is called a *polynomial* — meaning simply something with many terms. Many important polynomials are made up of a set of terms each of which contains a different power of a single quantity, x. We can then write:

$$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n,$$

where the quantities a_0, a_1, a_2, a_3, etc., are constant coefficients, labeled here to show which power of x they are associated with. We have written the combination as $P(x)$, to indicate that this combination, P, is a certain *function* of x if x is a mathematical variable. That is an aspect of polynomials which is important in calculus, but we won't expand on it in these algebra review notes.

A polynomial has a highest power of x in it; this is called the *degree* of the polynomial. Thus if the highest term is $3x^4$, we say that the polynomial is "a polynomial of degree four" or "a quartic." For the most part, we shall not go beyond quadratics (degree 2).

The basis of many polynomials is a binomial (two-term) combination of two variables, of the form $(x+y)$, raised to an arbitrary power. The simplest examples of this are $(x+y)$ itself and $(x+y)^2 = x^2 + 2xy + y^2$. A *binomial expansion* is the result of multiplying out such an expression as $(x+y)^n$ into a polynomial. The general *binomial theorem* states that

$$(a+b)^n = a^n + na^{n-1}b + \cdots + \binom{n}{k} a^{n-k} b^k + \cdots + nab^{n-1} + b^n$$

where the coefficient of the general term is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!},$$

where $n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$.

If we want to *add* one polynomial to another, we simply add the terms belonging to the same power of x.

Multiplying one polynomial by another is a bit more complicated, but again involves identifying all the terms that have the same power of x and adding the coefficients to make a single term of the form $a_n x^n$.

Example:
$$\begin{aligned}(x+2)(2x^2-3x+4) &= x(2x^2-3x+4)+2(2x^2-3x+4)\\ &= (2x^3-3x^2+4x)+(4x^2-6x+8)\\ &= 2x^3+(-3x^2+4x^2)+(4x-6x)+8\\ &= 2x^3+x^2-2x+8.\end{aligned}$$

Exercise 2.4.1:
Multiply $(3x^2+4x-5)$ by $(2x-1)$.

2.5 Quadratic Equations

A quadratic equation is a relationship that can be manipulated into the form:
$$ax^2+bx+c=0.$$

We can solve the equation through the process of *completing the square*. This technique is the basis of the general quadratic formula that you are likely familiar with already:
$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$
We'll give a specific example and then derive the general formula.

2.5.1 Completing the Square: An Example

Suppose we are asked to solve the equation:
$$x^2-6x-4=0$$
We recognize that the combination (x^2-6x) is the first two terms of $(x-3)^2$. We just evaluate the complete square and see what is left over:
$$\begin{aligned}(x-3)^2 &= x^2-6x+9\\ x^2-6x &= (x-3)^2-9\\ x^2-6x-4 &= \left((x-3)^2-9\right)-4\\ &= (x-3)^2-13.\end{aligned}$$

But $x^2 - 6x - 4 = 0 \Rightarrow (x-3)^2 - 13 = 0$, requiring $(x-3) = \pm\sqrt{13}$. Thus the solution is:

$$x = 3 \pm \sqrt{13}.$$

[**An important point:** Notice that, in this example, our original quadratic expression reduces to a perfect square *minus* a certain number (13). When we set the whole expression equal to zero, this means that the perfect square is equal to a *positive* number, and we can proceed to take the square root of both sides. But if the quadratic had been a perfect square *plus* some number, n, we would have arrived at an equation of the form:

$$(x+p)^2 = -n.$$

We should then have been faced with taking the square root of a negative number. That would take us into the realm of *imaginary numbers*. That is beyond the scope of anything you need consider for the present.]

2.5.2 The General Quadratic Formula

In general:

a) First rearrange the quadratic equation into the following form if it is not already in that form:

$$ax^2 + bx + c = 0.$$

b) Subtract the constant c from both sides:

$$ax^2 + bx = -c.$$

c) Divide through by the coefficient a:

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

d) Complete the square of the left-hand side by adding $(b/2a)^2$. Add the same quantity to the right side:

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2.$$

e) Bring the right-hand side to a common denominator and take the square root of both sides:

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}.$$

f) Subtract $\frac{b}{2a}$ from both sides:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

And there you have the general quadratic formula.

Notice that, if the solutions are to be real numbers, we must have $b^2 - 4ac \geq 0$.

Exercise 2.5.1:
Solve these quadratic equations by completing the square. (If the coefficient of x^2 is not 1, you should follow the steps listed above in the derivation of the general formula):

a) $x^2 - 4x - 12 = 0$

b) $9x^2 - 6x - 1 = 0$

c) $x + 2x^2 = \frac{5}{8}$

d) $(x + 2)(2x - 1) + 3(x + 1) = 4.$

Exercise 2.5.2:
Solve these quadratic equations by the general formula:

a) $2x^2 = 9x - 8$

b) Solve for t: $y = ut - \frac{1}{2}gt^2$

c) $5x(x + 2) = 2(1 - x)$

d) $0.2x^2 - 1.5x = 3$

e) Solve for x: $x^2 - 2sx = 1 - 2s^2$.

2.6 Long Division & Factoring

Dividing one polynomial by another is very much like long division in arithmetic. We shall assume that the polynomial we are dividing into (the *dividend*) is of a higher degree than the one we are dividing by (the *divisor*). The reason for doing the division may be to find out if the higher-degree polynomial is exactly divisible by the lower-degree one; if it is, we have identified a way of *factoring* the higher-degree polynomial. We write both dividend and divisor in decreasing powers of x, or whatever variable is being used.

Example: Divide $4x^2 + 8x + 3$ by $2x + 1$.

$$\begin{array}{r|l} & 2x+3 \\ \hline 2x+1 & 4x^2+8x+3 \\ & -4x^2-2x \\ \hline & 6x+3 \\ & -6x-3 \\ \hline & 0 \end{array}$$

It works! $2x + 1$ divides exactly into $4x^2 + 8x + 3$; the result is $(2x + 3)$, and so we have factored the quadratic into two linear factors.

Exercise 2.6.1:
Divide $5x^2 - 6x - 8$ by $x - 2$.

Let's now change the above example of long division slightly, by making our dividend $4x^2 + 8x + 5$. There will now be a remainder of 2.

The remainder 2 means that the quotient still has a fraction in it.

$$\frac{4x^2 + 8x + 5}{2x + 1} = 2x + 3 + \frac{2}{2x + 1}$$

This is analogous to the similar process with numbers; for instance,

$$\frac{423}{10} = 42 + \frac{3}{10}.$$

With numbers we know that the remainder is always a number less than the divisor when dividing by 10; the possible remainders are $0, 1, \cdots, 9$.

For polynomials, the remainder is always of *lower degree* than the divisor. For example:

$$\frac{a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0}{x^2+1} = Q(x) + \frac{R(x)}{x^2+1}.$$

The quotient $Q(x)$ has degree $3 = 5-2$ and the remainder $R(x)$ has degree at most 1. In other words, there are constants $b_0, b_1, b_2, b_3, c_0, c_1$ for which

$$Q(x) = b_3x^3 + b_2x^2 + b_1x + b_0 \text{ and } R(x) = c_1x + c_0$$

Exercise 2.6.2:
Divide $(2x^3 + 2x^2 - 10x + 4)$ by $(x+3)$. What is the degree of the remainder?

Exercise 2.6.3:
If the polynomial $(x^5 + 4x^4 - 6x^3 + 5x^2 - 2x + 3)$ were divided by (x^2+2) (*Don't* actually do it!) what would be the degree of the quotient and the degree of the remainder?

Exercise 2.6.4:
What happens if you divide a polynomial of lower degree by one of higher degree?

Factoring is an important part of many calculations. But it is often hard to see which polynomial evenly (exactly) divides another. (In fact, factoring polynomials and numbers is an important problem for both mathematicians and computer scientists.) Here is the only device you need to know for now:

If $P(r) = 0$, then $(x-r)$ divides $P(x)$ evenly.

This is what we mean by saying that r is a *root* of $P(x)$. Example: $P(x) = x^2 + x - 2$.

$P(x)$ has the root -2, because $P(-2) = (-2)^2 + (-2) - 2 = 0$. Then $P(x)$ is exactly divisible by $(x - (-2)) = x + 2$:

$$\begin{array}{r} x - 1 \\ x+2 \overline{) \; x^2 \; + x - 2} \\ \underline{- x^2 - 2x} \quad \\ - x - 2 \\ \underline{x + 2} \\ 0 \end{array}$$

The quotient is $x - 1$. Therefore, $x^2 + x - 2 = (x + 2)(x - 1)$.

(The actual process of division here is of course just like that in the example at the beginning of the section. But there we gave the value of the divisor, instead of looking for it from scratch.)

When dividing by a first order factor that divides evenly, you may save time by solving for coefficients using the following scheme:

$$\begin{aligned} \text{Put } x^2 + x - 2 &= (x + 2)(ax + b) \\ &= ax^2 + (2a + b)x + 2b \end{aligned}$$

Equating coefficients gives $a = 1, b = -1$. [But you can see in this case that $a = 1$ without writing down anything; then you can solve for b in $x^2 + x - 2 = (x + 2)(x + b)$.]

Here is another approach to factoring: For polynomials $x^n + a_{n-1}x^{n-1} + \cdots + a_0$ with leading coefficient 1, and integers for the other coefficients, any integer root must be a divisor of the constant term a_0. Thus in the polynomial $x^2 + x - 2$ we have $a_0 = -2$, and there are four integers that might work: ± 1 and ± 2.

Finally, for quadratic equations the roots are always obtainable by the quadratic formula:

$$x^2 + x - 2 = 0 \Rightarrow x = \frac{-1 \pm \sqrt{1^2 - 4(-2)}}{2} = \frac{-1 \pm 3}{2} = -2 \text{ or } 1.$$

In general, $ax^2 + bx + c = a(x - r_1)(x - r_2)$ where r_1 and r_2 are the roots. If the quadratic formula leads to the square root of a negative number than the quadratic has no real roots and no real factors. (If you allow the use of complex numbers as coefficients, then the quadratic factors as usual.)

[**Look out for easily factorable quadratics!** Be on the alert for the following:

- Quadratics whose coefficients are small or simple enough for you to make a shrewd guess at the factorization;
- Quadratics that are perfect squares;
- Expressions of the form $a^2x^2 - b^2$, which can be immediately factored into $(ax+b)(ax-b)$.]

Don't overuse the quadratic formula when factoring a polynomial like $x^2 + x - 2$. Your first instinct should be to check ± 1 and ± 2 as roots. If you do use the quadratic formula, you should be prepared to *double check* your answer. Arithmetic errors can be as significant in a physical problem as the difference between going the right way and the wrong way down a one-way street.

Exercise 2.6.5:
Test whether the following quadratics can be factored, and find the factors if they exist:

a) $8x^2 + 14x - 15$

b) $2x^2 - 3x + 10$

c) $9x^2 - 24ax + 16a^2$

d) $2x^2 + 10x - 56$

e) $100x^4 - 10^{10}$. (This is of 4th degree, but first put $x^2 = u$.)

Exercise 2.6.6:
Solve these quadratic equations by factoring:

a) $x^2 - 5x + 6 = 0$

b) $2x^2 - 5x - 12 = 0$

c) $3x^2 - 5x = 0$

d) $6x^2 - 7x - 20 = 0$

2.7 Algebraic Manipulations

2.7.1 Eliminating Radicals

(Mathematical, not political)

Radicals are a nuisance if one is trying to solve an algebraic equation, and one usually wants to get rid of them. The way to get rid of a radical is of course to square it (or cube for a cube root, etc.); but this only works if the radical stands by itself on one side of an equation. Thus if you have:

$$\sqrt{x-1} + a = b,$$

you don't gain anything by squaring both sides. That simply gives you $(x-1) + 2a\sqrt{x-1} + a^2 = b^2$.

If you first isolate the radical on the left-hand side, you have

$$\sqrt{x-1} = b - a.$$

Then when you square, you get: $x - 1 = (b-a)^2$, and you are in business.

You may have to go through this routine more than once.

Example: Suppose you have $\sqrt{2x-1} - 1 = \sqrt{x-1}$.

Now you can't isolate both radicals — and it doesn't help to put them together on one side of the equation. So you do the best you can, with one of the radicals isolated in the equation as it stands. Squaring, you get:

$$2x - 1 - 2\sqrt{2x-1} + 1 = x - 1.$$

Now we can isolate the remaining radical:

$$2\sqrt{2x-1} = x + 1.$$

Squaring again, and rearranging, gives us a quadratic equation:

$$x^2 - 6x + 5 = 0 \text{ leading to } x = 1 \text{ or } 5.$$

[**Warning!** Squaring may introduce so-called *extraneous* roots, because, for example, if we started with $x = 3$ and squared it, we would have $x^2 = 9$. Taking the square root of **this** would appear to allow $x = -3$ as well as $x = 3$. So always check your final results to see if they fit the equation in its original form.]

Exercise 2.7.1:
Solve for x:

a) $\sqrt{2x-1} = x - 2$ (Be careful!)

b) $\sqrt{x-3} = \sqrt{2x-5} - 1$

2.7.2 Combining Fractions

1. Multiplying: $\frac{a}{b}\frac{c}{d} = \frac{ac}{bd}$.

2. Dividing: $\frac{(a/b)}{(c/d)} = \frac{a}{b}\frac{d}{c} = \frac{ad}{bc}$.

3. Adding: $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$.

4. Subtracting: $\frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}$.

The quantities a, b, c, d may be ordinary numbers, but they may also be algebraic expressions. In either case, look first to see if they can be factored. In (1) and (2), this may lead to simple cancellation; in (3) and (4) it may enable you to find a least common denominator that is simpler than the product bd. In the latter case, convert each fraction to one with this common denominator and then add and subtract the numerators as required. This may involve less complexity than if you mechanically apply the expressions above.

Exercise 2.7.2:
Evaluate the following combinations of fractions, bringing the result to a single denominator in each case and canceling where possible:

a) $\left(\frac{x-y}{x^2+y^2}\right)\left(\frac{y}{y-x}\right)$

b) $\frac{\left(\frac{x^2y+y^3}{2x}\right)}{\left(\frac{2xy+y^2}{4x^3}\right)}$

c) $\frac{x}{4y^2x} + \frac{z^2}{8xy}$

d) $\frac{4z}{xy^2} - \frac{2x}{y^3x} + \frac{z}{x^2y}$

2.8 Geometric Series and Geometric Progressions

The formula for the sum of the *infinite geometric series* is:

$$1 + r + r^2 + r^3 + \cdots + r^n + \cdots = \frac{1}{1-r}$$

if $-1 < r < 1$.

If the series is stopped after n terms, we get what is called a *geometric sum* or *geometric progression*; its sum is given by

$$1 + r + r^2 + r^3 + \cdots + r^n = \frac{1-r^{n+1}}{1-r}.$$

In the above formulas, both sides can be multiplied by a constant factor a, giving a formula for the sum of $a + ar + ar^2 + \cdots$. It's easiest to remember the formulas in the above form, however. In fact, if you learn how to derive the second formula from the first you can get away with just remembering the formula for the sum of the series.

Exercise 2.8.1:
Express each of the following sums as a common fraction.

a) $1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \cdots$

b) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

c) $1 + .1 + .01 + \cdots$ (Hint: $1 + \frac{1}{10} + \frac{1}{100} + \cdots$)

2.9 Answers to Exercises

Answers to exercise 2.1.2 not provided.

Exercise 2.1.1: (a) $8.0516 \cdot 10^4$; (b) $7.51 \cdot 10^{-2}$; (c) $3.52 \cdot 10^6$; (d) $8.1 \cdot 10^{-8}$

Exercise 2.1.3: (a) 1.125; (b) $1.39 \cdot 10^4$; (c) $2 \cdot 10^6$

Exercise 2.1.4: (a) $1.051 \cdot 10^{10}$; (b) $4.65 \cdot 10^5$; (c) $-1.429 \cdot 10^{26}$; (d) $3.95 \cdot 10^{-19}$; (e) $6.72 \cdot 10^{-9}$

Exercise 2.3.1: (a) $x = \frac{-11}{24}$; (b) $x = \frac{30}{17}$; (c) $x = \frac{c-3b}{3a-5b}$

Exercise 2.3.2: (a) $x = \frac{7}{5}$, $y = -\frac{2}{5}$; (b) $a = -\frac{8}{9}$, $b = \frac{14}{9}$; (c) $x = 1$, $y = 2$

Exercise 2.4.1: $6x^3 + 5x^2 - 14x + 5$

Exercise 2.5.1: (a) $6, -2$; (b) $\frac{1\pm\sqrt{2}}{3}$; (c) $\frac{-1\pm\sqrt{6}}{4}$; (d) $\frac{-3\pm\sqrt{15}}{2}$

Exercise 2.5.2: (a) $\frac{9\pm\sqrt{17}}{4}$; (b) $\frac{u\pm\sqrt{u^2-2gy}}{g}$; (c) $\frac{-6\pm\sqrt{46}}{5}$; (d) $\frac{15\pm\sqrt{465}}{4}$; (e) $s \pm \sqrt{1-s^2}$

Exercise 2.6.1: $5x + 4$

Exercise 2.6.2: The degree of the remainder is zero. (The remainder is $-2 = -2x^0$.)

Exercise 2.6.3: Degree of quotient: 3; degree of remainder: 1 (at most).

Exercise 2.6.4: You can't do it by long division. We assumed that the divider had a lower degree than the dividend, in order for the long division to work. It's like trying to divide 2 by 7. So the answer is (again, without using long division!): quotient $= 0$; remainder = dividend.

Exercise 2.6.5: (a) $(4x-3)(2x+5)$; (b) No (complex roots); (c) $(3x - 4a)^2$; (d) $2\left(x + \frac{5}{2} - \frac{\sqrt{137}}{2}\right)\left(x + \frac{5}{2} + \frac{\sqrt{137}}{2}\right)$; (e) $100\left(x^2 + 10^4\right)(x+100)(x-100)$

Exercise 2.6.6: (a) $(x-2)(x-3)=0 \Rightarrow x=2,3$; (b) $(2x-3)(x+4)=0 \Rightarrow x=\frac{3}{2},-4$; (c) $x(3x-5)=0 \Rightarrow x=\frac{5}{3},0$; (d) $(2x-5)(3x+4)=0 \Rightarrow x=\frac{5}{2},-\frac{4}{3}$

Exercise 2.7.1: (a) $x=5$ ($x=1$ doesn't work, since in the original equation the LHS is a radical, and thus greater than 0, and the RHS is $1-2=-1<0$); (b) $x=3$ or 7

Exercise 2.7.2: (a) $\frac{-y}{x^2+y^2}$; (b) $\frac{2x^2\left(x^2+y^2\right)}{2x+y}$; (c) $\frac{2x^2+yz^3}{8xy^2z}$; (d) $\frac{4xyz^2-2x^3+y^2z^2}{x^2y^3z}$

Exercise 2.8.1: (a) $r=1/3$ so $\frac{1}{1-r}=3/2$; (b) $r=1/2$ and $\frac{1}{1-r}=2$; (c) Applying the hint, $r=1/10$ and $\frac{1}{1-r}=10/9$. Hence $10/9=1.111....$

This module is based in large part on an earlier module prepared by the Department of Mathematics.

3 Algebra Review Problems

3.1 Calculating

(See Sections 2.1 and 2.2 of the review module.)

Try a selection of the problems in this section. Give your answer in scientific notation, with the correct number of significant digits. No calculators!

Problem 1: Express as a single number, in scientific notation $a \times 10^k$, $1 \leq a < 10$:

a) $\frac{(.024)(3\times10^{-2})}{8\times10^{12}}$

b) $\frac{1600}{32\times10^{-4}}$

Problem 2: 6.25×10^{24} molecules of water fill a .2 liter glass. Approximately how much of this volume (in liters) does one molecule account for? (Give your answer in scientific notation with the correct number of significant figures.)

Problem 3: Using $E = mc^2$, where $c = 3 \times 10^8$ meters/sec, in km/sec units how much energy E is mass-equivalent to 3×10^{-18} kg of water?

Problem 4: Looking at a representative .60 ml sample under her microscope, Michelle counts exactly 120 bacteria. If the sample is drawn from a flask containing 2000. ml of water, approximately how many bacteria are in the flask?

3.2 Linear equations

(See Section 2.3 of the review module.)

Problem 5: Solve for x in terms of a and b: $\frac{x+a}{x-a} = b$.

Problem 6: Solve simultaneously for x and y:

a) $\begin{aligned} 2x - 3y &= -1 \\ 3x - 2y &= 6 \end{aligned}$ b) $\begin{aligned} 4x - 3y &= 9 \\ 5y - 3x - 7 &= 0 \end{aligned}$ c) $\begin{aligned} x + 2y &= a \\ x - y &= b \end{aligned}$

Problem 7: Brown rice comes in 5 lb. bags costing \$2 a bag; wild rice comes in 2 lb. bags costing \$10 a bag. Egbert has just spent \$32 buying 34 lbs. of rice for his commune. How many bags of each did he buy?

Problem 8: How many liters of liquid endersol should be added to 20 liters of water to get a solution that is 40% endersol?

3.3 Polynomials, binomial theorem

(See Section 2.4 of the review module.)

Problem 9: Write $(x+1)^3$ as a polynomial in x, by using the binomial theorem.

Problem 10: Simplify $\frac{(x+h)^3-x^3}{h}$, writing it as a polynomial in x and h.

Problem 11: What are the coefficients a, b, c in $(x+1)^{12} = ax^{12} + bx^{11} + cx^{10} + \cdots$?

Problem 12: Write $(x+y)^4$ as a polynomial in x and y.

3.4 Quadratic equations

(See Section 2.5 of the review module.)

Try a selection of these problems; similar problems are grouped together. If the coefficient of x^2 is not 1, be especially careful in using the quadratic formula.

Problem 13: For each of the following, first try to find the roots by factoring the left hand side. If successful, then use the quadratic formula to check your answer; if not, find the roots by the quadratic formula. Try a, c, and d, b for more practice.

a) $x^2 - 7x + 12 = 0$

b) $x^2 + 2x - 35 = 0$

c) $x^2 - 5x + 5 = 0$

d) $2x^2 - 5x + 3 = 0$

Problem 14:

a) Solve for t in terms of the other variables: $y = vt - \frac{1}{2}gt^2$.

b) Express x in terms of a, b, and h, if $h = 2ax - bx^2$ and $a, b, h, x > 0$.

Problem 15: Solve $x^2 + 3x - 1 = 0$, and tell whether both roots are positive, negative, or of opposite signs. How could you have answered this last question without first solving the equation?

Problem 16: Let a, b be the distinct roots of $x^2 - 4x + 2$, with $a < b$. Which correctly compares a and b to 1: $a < b < 1$, $a < 1 < b$, or $1 < a < b$?

Problem 17: For which values of the constant b will the roots be real:

a) $x^2 + 4x - b = 0$

b) $x^2 + bx + 1 = 0$

Problem 18: A rectangle has area 7, and the width is 2 less than the height. What are its dimensions?

Problem 19: The sum of three times a certain number and twice its reciprocal is 5. Find all such numbers.

Problem 20: A rock is thrown off the ledge of a 100 foot cliff at time $t = 0$. The height of the rock above the ground is then given in terms of time t (in seconds) by the formula $40t - 16t^2 + 100$. After how many seconds does the rock hit the ground?

3.5 Factoring

(See Section 2.6 of the review module.)

These depend on the factor theorem:

> $(x - k)$ is a factor of the polynomial $f(x) \Leftrightarrow k$ is a zero, i.e., $f(k) = 0$.

If the polynomial has leading coefficient 1 *and integer coefficients, its integer zeros will divide the constant term. Once you know a factor* $(x-k)$*, find the other factor by comparing coefficients, or use long division if you must.*

Problem 21: Factor each of the following, using any extra information that is given:

a) $f(x) = x^3 - 2x^2 - x + 2, \quad f(-1) = 0.$

b) $f(x) = x^3 - 2x - 4, \quad f(2) = 0.$

c) $x^4 - 16$

d) $x^3 - x^2 - 9x + 9$

Problem 22: Does $x - 1$ divide evenly the polynomial:

$$5x^8 - 3x^5 + x^4 - 2x - 1?$$

(Hint: do not attempt to actually do the division!)

Problem 23: For what value(s) of the constant c will:

a) $x - 1$ be a factor of $x^3 - 3x^2 + cx - 2$;

b) $x + 2$ be a factor of $x^4 + 4x^3 + c$?

3.6 Solving other types of equations; algebraic manipulations

(See Section 2.7 of the review module.)

Each of these is a little different from the others; try one from each group.

Problem 24: Solve for x:

a) $\sqrt{3x+10} = x + 2$

b) $\sqrt{1-x^2} = x/\sqrt{3}$

Problem 25: Express x in terms of y, if $\frac{1+x^2}{1-x^2} = y$.

Problem 26: Express m in terms of c and a: $(m^2 - c^2)^{-1/2} = a$, given that $m > 0$.

Problem 27: Combine the two terms:

a) $\frac{a}{a+h} + \frac{h}{a-h}$

b) $\frac{x}{x+1} + \frac{2}{x-2}$

Problem 28: Solve: $3(x+1)^{-1} + (x-3)^{-1} = 1$.

Problem 29: For each positive integer n, a_n is given; evaluate and simplify a_{n+1}/a_n:

a) $a_n = \frac{x^n}{n!}$

b) $a_n = \frac{x^{2n}}{n(n+1)}$

3.7 Geometric series and geometric progressions

(See Section 2.8 of the review module.)

Problem 30: Find the sum of:

a) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots$

b) $2 + \frac{2}{10} + \frac{2}{100} + \cdots + \frac{2}{10^5} + \cdots$

Problem 31: Express each of the following repeating decimals as a common fraction, by interpreting it as an infinite geometric series:

a) $0.44444444\ldots$

b) $0.12121212\ldots$

Problem 32: A turtle travels along a road. The first day it covers 100 feet, and on each succeeding day it covers 2/3 the distance it traveled the day before. How far can it get in this way?

Problem 33: Prove the formula for the sum of a geometric progression by cross-multiplying.

Problem 34: Derive the formula:

$$1 + r + r^2 + r^3 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

$(-1 < r < 1)$ by writing:

$$1 + r + \cdots + r^n = (1 + r + \cdots + r^n + \cdots) - r^{n+1}(1 + r + \cdots + r_n + \cdots).$$

3.8 Solutions

Solution 1:

a)

$$\begin{aligned}\frac{(.024)(3 \times 10^{-2})}{8 \times 10^{12}} &= \frac{24 \times 10^{-3} \times 3 \times 10^{-2}}{8 \times 10^{12}} \\ &= \left(\frac{24 \cdot 3}{8}\right) 10^{-5} \cdot 10^{-12} = \boxed{9 \times 10^{-17}}\end{aligned}$$

b)

$$\frac{1600}{32 \times 10^{-4}} = \frac{16 \times 10^2}{32 \times 10^{-4}} = .5 \times 10^6 = \boxed{5 \times 10^5}$$

Solution 2: Let x be the volume accounted for by one molecule. Then:

$$\begin{aligned}\frac{6.25 \times 10^{24}}{.2} &= \frac{1}{x} \\ x &= \frac{.2}{6.25 \times 10^{24}} = \frac{2}{6.25} \times \frac{10^{-1}}{10^{24}} \\ &= .3 \times 10^{-25} = \boxed{3 \times 10^{-26}} \text{ liters.}\end{aligned}$$

Solution 3: $E = (3 \times 10^{-18})(3 \times 10^8)^2 = 27 \times 10^{-18} \times 10^{16} = \boxed{2.7 \times 10^{-1}}$.

Solution 4: $\frac{.60}{120} = \frac{2000}{x} \Rightarrow x = \frac{2000}{.60} \times 120 = \frac{2 \times 12 \times 10^3 \times 10}{6 \times 10^{-1}} = \boxed{4 \times 10^5}$

Solution 5: Cross-multiply $\frac{x+a}{x-a} = b$ to get

$$\begin{aligned} x + a &= bx - ba \\ x - bx &= -a - ba \\ x &= -\frac{a(1+b)}{1-b} \\ x &= \boxed{\frac{a(b+1)}{b-1}} \end{aligned}$$

Solution 6:

a) We have

$$\begin{aligned} 2x - 3y &= -1 \\ 3x - 2y &= 6 \end{aligned}$$

To eliminate y, we multiply the top by 2 and bottom by 3 and subtract:

$$\begin{aligned} 4x - 6y &= -2 \\ 9x - 6y &= 18 \\ -5x &= -20 \end{aligned}$$

$\boxed{x = 4, y = 3}$

b) Given:

$$\begin{aligned} 4x - 3y &= 9 \\ -3x + 5y &= 7 \end{aligned}$$

Multiply top by 5, bottom by 3, and add: $\begin{array}{lcl} 20x - 15y & = & 45 \\ -9x + 15y & = & 21 \\ 11x & = & 66 \end{array}$ $\boxed{x = 6, y = 5}$

c)

$$\begin{aligned} x + 2y &= a \\ x - y &= b \\ 3y &= a - b \\ y &= \frac{a-b}{3} \end{aligned}$$

$x = \frac{a+2b}{3}, y = \frac{a-b}{3}$.

Solution 7: Let x be the number of bags of brown rice and y be the number of bags of wild rice; then we have the system of equations:

$$\begin{aligned} 2x + 10y &= 32 \text{ (dollars)} \\ 5x + 2y &= 34 \text{ (pounds)} \end{aligned}$$

Eliminate y by multiplying the bottom equation by 5 and subtracting: $-23x = -138 \Rightarrow \boxed{x = 6, y = 2.}$

Solution 8: x is the number of liters of endersol.

$$\frac{x}{x+20} = \frac{40}{100} = \frac{2}{5} \Rightarrow 5x = 2x + 40 \Rightarrow \boxed{x = 40/3 \text{ liters.}}$$

Solution 9:

$$\begin{aligned} (x+1)^3 &= x^3 + \binom{3}{1}x^2 + \binom{3}{2}x + \binom{3}{3} \\ &= x^3 + 3x^2 + 3x + 1 \end{aligned}$$

Solution 10:

$$\begin{aligned} \frac{(x+h)^3 - x^3}{h} &= \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\ &= 3x^2 + 3xh + h^2 \end{aligned}$$

Solution 11: Note that $\binom{12}{2} = \frac{12\cdot 11}{2} = 6 \cdot 11 = 66$.

$$\begin{aligned}(x+1)^{12} &= x^{12} + \binom{12}{1}x^{11} + \binom{12}{2}x^{10} + \cdots \\ &= x^{12} + 12x^{11} + 66x^{10} + \cdots\end{aligned}$$

Solution 12: $(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$
(Note: $\binom{4}{2} = \frac{4\cdot 3}{2} = 6$.)

Solution 13:

a) $x^2 - 7x + 12 = (x-3)(x-4) = 0 \Rightarrow \boxed{x = 3, 4}$
or $x = \frac{7\pm\sqrt{49-4\cdot 12}}{2} = \frac{7\pm 1}{2} = 3, 4$.

b) $x^2 + 2x - 35 = (x+7)(x-5) = 0 \Rightarrow \boxed{x = -7, 5}$
or $x = \frac{-2\pm\sqrt{4+4\cdot 35}}{2} = \frac{-2\pm 12}{2} = 7, 5$.

c) $x^2 - 5x + 5$ doesn't factor. By formula: $x = \frac{5\pm\sqrt{25+4\cdot 5}}{2} = \frac{5\pm\sqrt{5}}{2}$

d) $2x^2 - 5x + 3 = (2x-3)(x-1) = 0 \Rightarrow \boxed{x = 1, \frac{3}{2}}$
or $x = \frac{5\pm\sqrt{25-4\cdot 3\cdot 2}}{2\cdot 2} = \frac{5\pm 1}{4} = 1, \frac{3}{2}$.

Solution 14:

a) $y = vt - \frac{1}{2}gt^2$ is a quadratic in t.

Rewrite as $\frac{1}{2}gt^2 - vt + y = 0$; by formula, $t = \frac{v\pm\sqrt{v^2-4\cdot\frac{g}{2}y}}{2\cdot g/2} = \frac{v\pm\sqrt{v^2-2gy}}{g}$.

b) $h = 2ax - bx^2$ is a quadratic in x; rewrite as $bx^2 - 2ax + h = 0$. By the formula, $x = \frac{2a\pm\sqrt{4a^2-4bh}}{2b} = \frac{a\pm\sqrt{a^2-bh}}{b}$.

Solution 15: $x^2 + 3x - 1 = 0$, so $x = \frac{-3\pm\sqrt{9-4(-1)}}{2} = \boxed{\dfrac{-3\pm\sqrt{13}}{2}}$.

Since $3 < \sqrt{13}$, we see that $-3+\sqrt{13} > 0$, while $-3-\sqrt{13} < 0$. Therefore, the roots are of *opposite* signs.

Alternately, since $x^2+3x-1 = (x-r_1)(x-r_2)$, we have $r_1 r_2 = -1$, which shows that roots have opposite signs without actually calculating them.

Solution 16: $x^2-4x+2=0$. By formula: $x = \frac{-4\pm\sqrt{16-4\cdot 2}}{2} = 2\pm\sqrt{2}$. Since $\sqrt{2}\approx 1.4$, $2-\sqrt{2} < 1 < 2+\sqrt{2}$.

Solution 17:

a) $x^2+4x-b=0$. The formula gives $x=\frac{-4\pm\sqrt{16+4b}}{2}$, so roots are real if $16+4b\geq 0$; i.e. $\boxed{b\geq -4.}$

b) $x^2+bx+1=0$. The formula gives $x=\frac{-b\pm\sqrt{b^2-4}}{2}$, so the roots are real if $b^2-4\geq 0$. But $b^2-4\geq 0$ implies $b^2\geq 4$, which implies $b\geq 2$ or $b\leq -2$.

Solution 18: Let x be the width and y be the height. We have

$$\begin{aligned} y-2 &= x \\ xy &= 7. \end{aligned}$$

Substitution into the second equation gives :

$$\begin{aligned} (y-2)y &= 7 \\ y^2-2y-7 &= 0 \\ \text{width } y &= \frac{2\pm\sqrt{4+28}}{2} = 2\sqrt{2}+1 \\ \text{length (height) } x &= 2\sqrt{2}-1 \end{aligned}$$

Solution 19: Let x be the number. Then $3x+\frac{2}{x}=5$ implies $3x^2-5x+2=0$. By the quadratic formula,

$$x = \frac{5\pm\sqrt{25-4\cdot 2\cdot 3}}{6} = \frac{5\pm 1}{6} = \boxed{1, \frac{2}{3}.}$$

Solution 20: Solve $16t^2 - 40t - 100 = 0$, or $4t^2 - 10t - 25 = 0$: this gives $t = \frac{10 \pm \sqrt{100+4\cdot 25\cdot 4}}{2\cdot 4} = \frac{10\pm 10\sqrt{5}}{8} = \boxed{\frac{5}{4}\left(1+\sqrt{5}\right)}$. We reject the negative solution because it makes no physical sense.

Solution 21:

a) $f(-1) = 0$ means that $x+1$ is a factor. We can then use long division to determine:

$$\begin{aligned} x^3 - 2x^2 - x + 2 &= (x+1)(x^2 - 3x + 2) \\ &= (x+1)(x-2)(x-1). \end{aligned}$$

b) $f(2) = 0$ means that $x - 2$ is a factor:

$$x^3 - 2x - 4 = (x-2)(x^2 + 2x + 2).$$

c) $x^4 - 16 = (x^2 - 4)(x^2 + 4) = (x-2)(x+2)(x^2+4)$.

d) For $x^3 - x^2 - 9x + 9$ we try as roots the factors of 9, and see that 1 is a root, so $x - 1$ is a factor.

$$\begin{aligned} x^3 - x^2 - 9x + 9 &= (x-1)(x^2 - 9) \\ &= (x-1)(x-3)(x+3). \end{aligned}$$

Solution 22: $x - 1$ divides $f(x)$ if and only if $f(1) = 0$. Because

$$f(1) = 5\cdot 1^8 - 3\cdot 1^5 + 1^4 - 2\cdot 1 - 1 = 0,$$

$x - 1$ is a factor and the answer is *yes*.

Solution 23:

a) $x - 1$ is a factor if and only if $f(1) = 0$, if and only if:

$$1^3 - 3\cdot 1^2 + c\cdot 1 - 2 = 0,$$

so we must have $c - 4 = 0$ or $\boxed{c = 4}$.

b) $x+2$ is a factor if and only if $f(-2)=0$, if and only if:

$$(-2)^4+4(-2)^3+c=0.$$

Hence we conclude that $16-32+c=0$ or $\boxed{c=16}$.

Solution 24:

a)

$$\begin{aligned}\sqrt{3x+10} &= x+2\\ 3x+10 &= x^2+4x+4\\ x^2+x-6 &= 0\\ (x+3)(x-2) &= 0\end{aligned}$$

We reject $x=-3$, since it doesn't satisfy the equation we started with; thus the only solution is $\boxed{x=2}$.

b)

$$\begin{aligned}\sqrt{1-x^2} &= \frac{x}{\sqrt{3}}\\ 1-x^2 &= \frac{x^2}{3}\\ 1 &= \frac{4}{3}x^2\\ x &= \sqrt{\frac{3}{4}} = \boxed{\frac{\sqrt{3}}{2}}\end{aligned}$$

Again, we reject $-\sqrt{3}/2$ because it doesn't satisfy the original equation.

Solution 25:

$$\begin{aligned}\frac{1+x^2}{1-x^2} &= y \quad \text{solve for } x^2\\ 1+x^2 &= y-x^2y\\ x^2(1+y) &= y-1\\ x^2 &= \frac{y-1}{y+1}, \text{ so}\end{aligned}$$

$$\boxed{x = \pm\sqrt{\frac{y-1}{y+1}}}$$

Solution 26:

$$\begin{aligned}
\frac{1}{\sqrt{m^2-c^2}} &= a \\
\frac{1}{m^2-c^2} &= a^2 \\
\frac{1}{a^2} &= m^2-c^2 \\
m^2 &= c^2+\frac{1}{a^2} \\
m &= \sqrt{c^2+\frac{1}{a^2}}
\end{aligned}$$

Solution 27:

a)

$$\frac{a}{a+h}+\frac{h}{a-h}=\frac{a(a-h)+h(a+h)}{(a+h)(a-h)}=\frac{a^2+h^2}{a^2-h^2}$$

b)

$$\frac{x}{x+1}+\frac{2}{x-2}=\frac{x^2+2}{(x+1)(x-2)}$$

Solution 28:

$$\begin{aligned}
\frac{3}{x+1}+\frac{1}{x-3} &= 1 \qquad \text{combine left-hand side} \\
\frac{4x-8}{x^2-2x-3} &= 1 \\
4x-8 &= x^2-2x-3 \\
x^2-6x+5 &= 0 \\
x-5)(x-1) &= 0 \\
x &= 1,5
\end{aligned}$$

Solution 29:

a) $\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} = \boxed{\dfrac{x}{n+1}}$

b) $\frac{a_{n+1}}{a_n} = \frac{x^{2(n+1)}}{(n+1)(n+2)} \cdot \frac{n(n+1)}{x^{2n}} = \boxed{x^2 \cdot \dfrac{n}{n+2}}$

Solution 30:

a) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots = \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{1}{3/2} = \boxed{\dfrac{2}{3}} \quad (r = -\frac{1}{2})$

b) $2 + \frac{2}{10} + \frac{2}{100} + \cdots = 2 \cdot \frac{1}{1-\frac{1}{10}} = \boxed{\dfrac{20}{9}} \quad (r = \frac{1}{10})$

Solution 31:

a) $.4444\ldots = \frac{4}{10} + \frac{4}{100} + \frac{4}{1000} + \ldots$
$= \frac{4}{10}\left(1 + \frac{1}{10} + \frac{1}{100} + \cdots\right) = \frac{4}{10}\left(\frac{1}{1-\frac{1}{10}}\right) = \boxed{\dfrac{4}{9}}$

b) $.121212\ldots = \frac{12}{100} + \frac{12}{10^4} + \frac{12}{10^6} + \ldots$
$= \frac{12}{100} \cdot \frac{1}{1-1/100} = \frac{12}{100} \cdot \frac{100}{99} = \boxed{\dfrac{12}{99}}$

Solution 32:

$$100 + \frac{2}{3}100 + \frac{2}{3} \cdot \frac{2}{3} \cdot 100 + \cdots = 100\left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots\right)$$
$$= 100 \cdot \frac{1}{1-2/3} = \boxed{300}$$

Solution 33:

$$(1 + r + \cdots + r^n)(1-r) = (1 + r + \cdots + r^n) - (r + r^2 + \cdots + r^{n+1})$$
$$= 1 - r^{n+1}$$

Solution 34:

$$\begin{aligned} 1 + r + \cdots + r^n &= (1 + r + r^2 + \cdots) - r^{n+1}(1 + r + \cdots) \\ &= \frac{1}{1-r} - \frac{r^{n+1}}{1-r} \\ &= \frac{1 - r^{n+1}}{1-r} \end{aligned}$$

4 Algebra Self-Tests

4.1 Algebra Diagnostic Exam #1

Problem 35: Evaluate and express in scientific notation with the correct number of significant figures:

$$\frac{1}{8 \cdot 10^{12}} \quad \text{and} \quad \frac{160}{32 \cdot 10^{-4}}.$$

Problem 36: Jack and Janet are buying an assortment of fruit. Jack wants the number of apples to be three less than twice the number of oranges. Janet wants to buy two more apples than oranges. How many oranges and apples should they get?

Problem 37: Expand the following polynomial and determine its degree:

$$x^4 - 3x^3 + 3x^2 + (2 - x^2)(x^2 + 2).$$

Problem 38: A man throws a rock off the ledge of a 100 foot high cliff at time $t = 0$. The height of the rock above ground level is then given in terms of the time t (seconds) by $40t - 16t^2 + 100$. After how many seconds does the rock hit the ground?

Problem 39: Factor $x^3 - x^2 - 16x + 16$.

Problem 40: Simplify $\frac{(x+h)^3 - x^3}{h}$.

4.2 Algebra Diagnostic Exam #1 Solutions

Solution 35: Evaluate and express in scientific notation with the correct number of significant figures:

$$\frac{1}{8 \cdot 10^{12}} \text{ and } \frac{160}{32 \cdot 10^{-4}}.$$

$$\frac{1}{8 \cdot 10^{12}} = \frac{1}{8} \times 10^{-12} = .125 \times 10^{-12} = \boxed{1.3 \times 10^{-13}}$$
$$\frac{160}{.32 \times 10^{-4}} = \frac{16 \times 10^{1}}{32 \times 10^{-6}} = .5 \times 10^{7} = \boxed{5 \times 10^{6}}$$

Solution 36: Jack and Janet are buying an assortment of fruit. Jack wants the number of apples to be three less than twice the number of oranges. Janet wants to buy two more apples than oranges. How many oranges and apples should they get?

Let x be the number of apples and y be the number of oranges. Then we have the system of equations:

$$x = 2y - 3$$
$$x = y + 2$$

Solve simultaneously. Subtracting, $0 = y - 5$ so $y = 5$ and $x = 7$. Answer: $\boxed{5 \text{ oranges}, 7 \text{ apples}}$.

Solution 37: Expand the following polynomial and determine its degree:

$$x^4 - 3x^3 + 3x^2 + (2 - x^2)(x^2 + 2)$$

$$= x^4 - 3x^3 + 3x^2 + (2x^2 + 4 - x^4 - 2x^2)$$
$$= -3x^3 + 3x^2 + 4 \qquad \boxed{\text{degree } 3}$$

Solution 38: A man throws a rock of the ledge of a 100 foot high cliff at time $t = 0$. The height of the rock above ground level is then given in terms of the time t (seconds) by $40t - 16t^2 + 100$. After how many seconds does the rock hit the ground?

The problem states the height at time t is $40t - 16t^2 + 100$, and asks what is t when the height is 0? Thus, we solve:

$$\begin{aligned} -16t^2 + 40t + 100 &= 0 \\ 4t^2 - 10t - 25 &= 0. \end{aligned}$$

This gives:

$$t = \frac{10 \pm \sqrt{100 - 4 \cdot 4 \cdot (-25)}}{2 \cdot 4} = \frac{10 \pm \sqrt{500}}{8}$$

and we reject the negative root to get $t = \frac{10}{8}\left(1 + \sqrt{5}\right) = \boxed{\frac{5}{4}\left(1 + \sqrt{5}\right).}$

Solution 39: Factor $x^3 - x^2 - 16x + 16$.

$f(x) = x^3 - x^2 - 16x + 16$ By inspection, $f(1) = 0$, so $x - 1$ is a factor.
$f(x) = (x-1)(x^2 - 16)$ by division or inspection)$f(x)$ $= \boxed{(x-1)(x-4)(x}$

Solution 40: Simplify $\frac{(x+h)^3 - x^3}{h}$.

$$\begin{aligned} \frac{(x+h)^3 - x^3}{h} &= \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\ &= \boxed{3x^2 + 3xh + h^2} \end{aligned}$$

4.3 Algebra Diagnostic Exam #2

Problem 41: 6.25×10^{24} molecules of water fill a .2 liter glass. Approximately how much of this volume (in liters) does one molecule account for?

Problem 42: Solve for x and y in terms of a and b, simplifying your answer as much as possible:

$$\begin{aligned} x + 2y &= a \\ x - y &= b \end{aligned}$$

Problem 43: Write $(x+1)^3$ as a polynomial in x.

Problem 44: Suppose a and b are the distinct roots of $x^2 - 4x - 2$, with $a < b$. Which of the following correctly compares a and b to 1?

i. $a < b < 1$

ii. $a < 1 < b$

iii. $a < b < 1$

Problem 45: Does $x - 1$ divide evenly into $5x^8 - 3x^5 + x^4 - 2x - 1$? (Hint: do not attempt to divide.)

Problem 46: First solve for x^2, then for x:

$$\frac{1 + x^2}{1 - x^2} = y$$

4.4 Algebra Diagnostic Exam #2 Solutions

Solution 41: 6.25×10^{24} molecules of water fill a .2 liter glass. Approximately how much of this volume (in liters) does one molecule account for?

$$\frac{.2}{6.25 \times 10^{24}} = \frac{20 \times 10^{-2}}{6.25 \times 10^{24}} = 3 \times 10^{-26} \text{ liters}$$

(to one significant figure.)

Solution 42: Solve for x and y in terms of a and b, simplifying your answer as much as possible:

$$\begin{aligned} x + 2y &= a \\ x - y &= b \end{aligned}$$

Subtracting: $3y = a - b$, so $\boxed{y = \frac{a-b}{3}}$.

Substituting: $x = y + b$, so $\boxed{x = \frac{a+2b}{3}}$.

Solution 43: Write $(x+1)^3$ as a polynomial in x.
$(x+1)^3 = x^3 + 3x^2 + 3x + 1$ by the binomial theorem.

Solution 44: Suppose a and b are the distinct roots of $x^2 - 4x - 2$, with $a < b$. Which of the following correctly compares a and b to 1?

i. $a < b < 1$

ii. $a < 1 < b$

iii. $a < b < 1$

The solutions to (roots of) $x^2 - 4x + 2 = 0$ are:

$$x = \frac{4 \pm \sqrt{16 - 4 \cdot 2}}{2} = \frac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2}.$$

Thus $a = 2 - \sqrt{2} \approx .6$, and $b = 2 + \sqrt{2} \approx 3.4$, so the correct comparison is $\boxed{a < 1 < b}$.

Solution 45: Does $x - 1$ divide evenly into $5x^8 - 3x^5 + x^4 - 2x - 1$? (Hint: do not attempt to divide.)

$x - 1$ is a factor of $f(x)$ if and only if $f(1) = 0$. But evaluating $f(1)$ gives us $5 - 3 + 1 - 2 - 1 = 0$, so $x - 1$ is a factor.

Solution 46: First solve for x^2, then for x:

$$\frac{1+x^2}{1-x^2} = y$$

Solving:

$$\begin{aligned} 1 + x^2 &= y - x^2 y \\ x^2(y+1) &= y - 1 \\ x^2 &= \frac{y-1}{y+1} \\ x &= \pm\sqrt{\frac{y-1}{y+1}} \end{aligned}$$

5 Algebra Self-Evaluation

You may want to informally evaluate your understanding of the various topic areas you have worked through in the *Self-Paced Review*. If you meet with tutors, you can show this evaluation to them and discuss whether you were accurate in your self-assessment.

For each topic which you have covered, grade yourself on a one to ten scale. One means you completely understand the topic and are able to solve all the problems without any hesitation. Ten means you could not solve any problems easily without review.

1.	Scientific Notation	____
2.	Significant Figures	____
3.1	Linear Equations in One Variable	____
3.2	Simultaneous Linear Equations in Two Variables	____
4.1	Polynomials	____
4.2	Binomial Theorem	____
5.	Quadratic Equations	____
6.1	Factoring	____
6.2	Long Division	____
7.1	Algebraic Manipulations: Eliminating Radicals	____
7.2	Algebraic Manipulations: Combining Fractions	____
8.1	Geometric Series	____
8.2	Geometric Progressions	____