Self-Paced Study Guide in Exponentials and Logarithms

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Contents

1 How to Use the Self-Paced Review Module

The *Self-Paced Review* consists of review modules with exercises; problems and solutions; self-tests and solutions; and self-evaluations for the four topic areas Algebra, Geometry and Analytic Geometry, Trigonometry, and Exponentials & Logarithms. In addition, previous *Diagnostic Exams* with solutions are included. Each topic area is independent of the others.

The *Review Modules* are designed to introduce the core material for each topic area. A numbering system facilitates easy tracking of subject material. For example, in the topic area Algebra, the subtopic Linear Equations is numbered with 2.3. Problems and self-evaluations are categorized using this numbering system.

When using the *Self-Paced Review*, it is important to differentiate between concept learning and problem solving. The review modules are oriented toward refreshing concept understanding while the problems and self-tests are designed to develop problem solving ability. When reviewing the modules, exercises are liberally sprinkled throughout the modules and should be solved while working through the module. The problems should be attempted without looking at the solutions. If a problem cannot be solved after at least two honest efforts, then consult the solutions. The tests should be taken only when both an understanding of the material and a problem solving ability have been achieved. The self-evaluation is a useful tool to evaluate one's mastery of the material. The previous Diagnostic Exams should provide the finishing touch.

The review modules were written by Professor A. P. French (Physics Department) and Adeliada Moranescu (MIT Class of 1994). The problems and solutions were written by Professor Arthur Mattuck (Mathematics Department). This document was originally produced by the Undergraduate Academic Affairs Office, August, 1992, and transcribed to LaTeX for OCW by Tea Dorminy (MIT Class of 2013) in August, 2010.

2 Exponentials & Logarithms Review Module

Exponentials and logarithms could well have been included within the Algebra module, since they are basically just part of the business of dealing with powers of numbers or powers of algebraic quantities. But they have so much importance in their own right that it is convenient to give them a module of their own.

2.1 Exponents

2.1.1 The Laws of Exponents

The concept of exponent begins with the multiplication of a given quantity a by itself an arbitrary number of times:

$$a^m = \underbrace{a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdots}_{m \text{ times}}$$

On the left we have the product expressed in exponential notation, which is very compact and efficient. The expression above is in effect a definition of what we mean by an exponent.

If we multiply a by itself a total of $(m+n)$ times, we can of course write it as the m-fold product multiplied by the n-fold product:

$$\underbrace{a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdots}_{(m+n) \text{ times}} = (\underbrace{a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdots}_{m \text{ times}})(\underbrace{a \cdot a \cdot a \cdot a \cdot a \cdots}_{n \text{ times}})$$

This is far more easily expressed in exponential form, and gives us the first rule for dealing with quantities expressed in this way:

$$(a^m)(a^n) = a^{m+n}.$$

When two quantities, written as powers of a given number, are multiplied together, we add the exponents.

Note that if we took the quantity a^m and multiplied it by itself p times, this would be the same as

(i) raising the quantity a^m to the pth power;

(ii) or multiplying a by itself mp times.

Therefore,

$$(a^m)^p = a^{mp}.$$

When a quantity, written as a power of a given number, is itself raised to a certain power, the exponents multiply.

Be sure you keep clear in your mind the distinction between this formula and the previous one; the difference can be enormous if large powers are involved. Take, for instance, the following quantities:

$$(10^3)(10^6) = (10)^{3+6} = 10^9 \qquad \text{one billion.}$$

But $(10^3)^6 = (10^3)(10^3)(10^3)(10^3)(10^3)(10^3) = 10^{18}$ a billion billion!

If we take a^m (the number a multiplied by itself m times) and *divide* it by a^n (a multiplied by itself n times), the result is a multiplied by itself $(m-n)$ times. Thus we have the rule for dividing one exponential by another:

$$\frac{a^m}{a^n} = a^{m-n}.$$

We can see from this that the reciprocal of any positive power of a is an equal negative power:

$$\frac{1}{a^n} = a^{-n}.$$

Also, if we put $m = n$ in the previous expression, the left-hand side is equal to 1. The right hand side is a^0. Thus, we have another result:

Any number (other than zero itself) raised to the power zero is equal to 1.

Exercise 2.1.1:
(No calculators!) Evaluate

a) $(2^4)(2^3)$

b) $(10^2)(10^4)$

c) $(10^2)^4$ [Compare with (b)]

d) $(2^5)(3^{-3})$

e) $(2^{-3})/(10^2)$

Exercise 2.1.2:
(No calculators!) Solve for x:

a) $x^5 = 32$

b) $2^x = 1$

c) $x^{-4} = 2$

d) $10^x = 0.000\,000\,000\,1$

NOTE: The answers to the exercises are all collected together at the end of this module. We have tried to eliminate errors, but if you find anything that you think needs to be corrected, please write to us.

2.1.2 Fractional Exponents

Go back to the expression for raising a quantity to a certain power and then raising the resulting number to some other power:

$$(a^m)^p = a^{mp}$$

If the product $mp = 1$, the right-hand side is just $a^1 = a$.

Suppose p is some specific integer, n. Then what the above equation says is that the parenthetical expression, raised to the nth power, is equal to a. But this means that the parenthetical expression is what we define as the *nth root of a*. Also, since $mp = mn = 1$, we must have $m = 1/n$. Thus:

A fractional power corresponds to taking roots of numbers:

$$a^{1/p} = \sqrt[p]{a}.$$

We can proceed from this to consider a wider variety of exponentials:

1) Suppose we take the *nth root* of a and raise it to the power m. Expressed in exponential form, this can be written:

$$\left(\sqrt[n]{a}\right)^m = a^{(m/n)}.$$

Thus we have a very convenient notation for writing any power of any root of a number.

2) The notation extends to negative powers also:

$$\frac{1}{\left(\sqrt[n]{a}\right)^m} = a^{-(m/n)}.$$

[Considering exponents as formed from products or ratios of integers is enough for practical calculations, since these use only finite decimals, which are rational numbers. (For example, $1.032 = 1032/1000$.) For non-rational values of exponents, *limits* are used: $\sqrt{2} = 1.4142\cdots$, so $3^{\sqrt{2}}$ is the limit of 3^1, $3^{1.4}$, $3^{1.41}$, $3^{1.414}$, $3^{1.4142}$, ...]

Exercise 2.1.3:
Simplify and evaluate:

a) $\left(2^8\right)^{1/2}$

b) $\left(\sqrt{0.0016}\right)^{-3}$

c) $\left(\frac{8}{27}\right)^{\frac{2}{3}}$

d) $\left(4^{1/3}\right)^{\frac{9}{2}}$

e) $100^{5/2}$

EXPONENTIALS & ROOTS: SUMMARY

$a^{m+n} = a^m a^n$	$a^{1/n} = \sqrt[n]{a}$ (n a positive integer)
$a^{-n} = 1/a^n$	$(a^m)^n = a^m n$
$a^0 = 1$	$a^m = a^n \rightarrow m = n$ (if $a \neq 1$)

2.2 Exponentials as Functions

Using limits, as above, we can take the exponent to be any number we please, not just a rational number or fraction. In other words, we arrive at the concept that, in a quantity written as a^x, a can be any chosen number and x can be a continuous variable taking on any possible value from $-\infty$ to ∞. Thus a^x becomes a continuous function of x.

$$y(x) = a^x.$$

> This is an *exponential function*. A fixed (given) number (a) is raised to an arbitrary power (x). The quantity a is the *base* of the exponential function.

[Contrast this with the function x^n. Here a continuously variable number (x) is raised to a definite given power (n).]

Everyday life provides what is probably the most familiar example of an exponential function: the growth of a savings account with a fixed compound interest rate. If, for example, the interest is compounded annually at a rate of 5%, then the amount A in the account after n years per dollar of initial deposit, is given by:

$$A(n) = (1.05)^n$$

However, in this computer age, interest may vary daily. If we again assume a 5% annual rate, the daily interest earned by $1 is 0.05/365, which is 0.00013698.... The interst is added to your initial dollar, yielding $1.00013698\cdots$ — almost insignificantly different from 1. In that case, after, say, 200 days, your initial dollar will be worth $(1.00013698...)^{200} \cong$ $1.0277, a gain of almost 2.8 cents. (Check all of this on your calculator.) If you deposit $600, the calculation above is done for each dollar in the deposit, so you end up with a total sum of $600(1.00013698...)^{200} =$ $616.66.

Exercise 2.2.1:
You put $200 into a savings account.

a) If the interest rate is 8% compounded annually, how many years will it take for your account to reach $250?

b) At the same annual rate, but compounded daily, what would be the balance in your account 500 days after the initial deposit?

The concept of the exponential function allows us to extend the range of quantities used as exponents. Besides being ordinary numbers, exponents can be expressions involving variables that can be manupulated in the same way as numbers.
Examples:

$$2^x 2^{-2x} = 2^{-x}; \qquad (10^{3x})^{1/x} = 10^3 = 1000.$$

Equations with the unknown in the exponent can be solved:
example:

$$\text{If } 2^x = 4^{1/x}, \text{ then } 2^x = (2^2)^{1/x} = 2^{2/x}, \text{ giving } x = 2/x \text{ and so } x = \pm\sqrt{2}.$$

Watch the exponents! (This is an extension of what we said about integral exponents in Section 2.1.1.) It is important not to get confused when you see a compound exponent. The notation should make things clear. Consider the following two cases:

> If you see $(10^x)^2$, you should read this as $(10^x)(10^x) = 10^{2x}$.
>
> But if you see 10^{x^2}, you should read this as $10^{xx} = (10^x)^x$.

If, for example, you put $x = 5$, the first expression is equal to 10^{10}, which is a big number; but the second expression is equal to 10^{25}, which is 15 orders of magnitude bigger.

Exercise 2.2.2:
Combine and simplify:

a) $7^w 7^{2w}$

b) $(3 \cdot 5^y)(5 \cdot 3^y)$

c) $(2^4)^2$

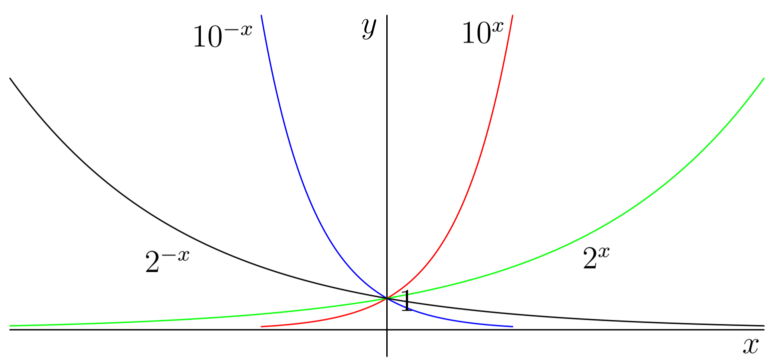
d) $16^a / 2^b$

2.3 Graphs of the Exponential Functions

In mathematics and science, although the base a of the exponential funciton could in principle be any number, there are only three values of it that you will need to worry about for most purposes:

$a = 2$	This is the basis of binary algebra, as used in computer science, etc.
$a = 10$	This is the basis of many other scientific calculations.
$a = e$	The symbol e stands for a special irrational number whose first ten digits are 2.718281828. It is central to the use of exponential functions in calculus, but we will not consider it further here. However, if you have already studied some calculus, you will very likely have met it.

The graphs below show the general appearance of the exponential functions 2^x, 10^x, and their reciprocals $(1/2)^x = 2^{-x}$ and $(1/10)^x = 10^{-x}$. All exponential functions are equal to 1 at $x = 0$. To describe an exponential that has some specific value other than 1 at $x = 0$, we simply put this value, call it $y(0)$, in front as a multiplying or scaling factor.



Notice that, with an exponential function, the factor of change for a given change of x is independent of where you start – the initial value (x_1) of x; it depends only on the difference $(x_2 - x_1)$ between initial and final values:

$$\text{If } y(x_1) = a^{x_1}, \text{ and } y(x_2) = a^{x_2}, \text{ then } \frac{y(x_1)}{y(x_2)} = a^{x_2 - x_1}.$$

2.4 Applications of Exponentials

Exponentials show up in all sorts of contexts. Here are a few examples:

2.4.1 Positive Exponential Examples

Compound interest: As already discussed,

$$A(t) = A(0)(1 + c)^t,$$

where c is the compound interest rate per unit of time and t is the time measured in those same units. For instance, if the rate is 5% compounded annually, then $c = 0.05$ and t is the time in years.

Growth of a biological population: A colony of bacteria, for example, grows by successive division, and may double in a few hours. One can put:

$$N(n) = N(0)2^n,$$

where n is the number of doubling times (τ) since the population was equal to $N(0)$ — i.e., $n = t/\tau$.

2.4.2 Negative Exponential Examples

Radioactive decay: This, like biological growth, can be described in terms of the time to produce a factor 2 of change – but in this case a factor 2 decrease. This time is the *half-life*, $t_{1/2}$, and one has

$$N(t) = N(0)\left(\frac{1}{2}\right)^{\frac{t}{t_{1/2}}} = N(0)2^{-t/t_{1/2}}.$$

Exercise 2.4.1:

a) A colony of bacteria in a test tube doubles every hour. If there are 2500 bacteria in the tube when the experimenter leaves for lunch at 12:00 noon, how many are there when she comes back at 1:30 PM? How many are there at 5:00 PM?

b) Find the half-life of a radioactive substance that has been left in a container for 6 days and decayed by a factor of 8.

2.5 Logarithms

Definition: If b is a fixed positive number (other than 1!), and if two other positive numbers x and y are related by:

$$y = b^x,$$

we say that *x is the logarithm of y to base b*, and we write this relation in the form:

$$x = \log_b y.$$

This means that, for any (positive) number, N, and for any base, b, the following relation always holds:

$$N = b^{\log_b N}.$$

(The word logarithm is too much a mouthful to use over and over again, so it is universally abbreviated as "log," not only in the above formula but also in speech.)

Below are some exercises based on this definition:

Exercise 2.5.1:
Find:

a) $\log_{10} 1000$

b) $\log_2 32$

c) $\log_2(1/16)$

d) $\log_b 1$

e) $\log_3 27$

f) $\log_2(-4)$

That last set of exercise illustrates the following points:

1) Logarithms of numbers greater than 1 are positive;

2) Logarithms of numbers between 0 and 1 exclusive are negative;

3) The logarithm of 1 is zero in any base;

4) You can't have a logarithm of a negative number (until you get to complex numbers).

This last result follows from the definition of a logarithm. If $y = b^x$, with b a positive number, then y is positive; its smallest possible value is 0, when $x = -\infty$. In other words, $\log_b 0 = -\infty$, regardless of the value of the base b.

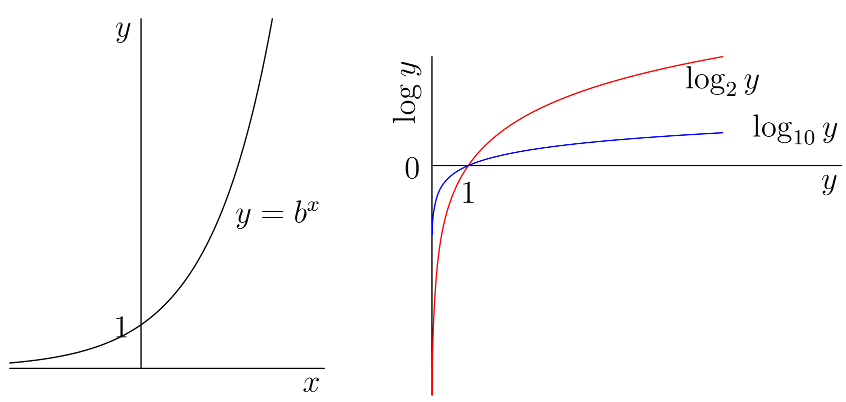
[*Logarithms to Base e*: Although we are not using the number e as the base of logarithms here, we should draw attention to the fact that the symbol for such logarithms is written as "ln" (for "natural logarithm") without any explicit definition of the base:

$$\ln x = \log_e x.$$

It is universally understood that any logarithm written in this way is to base e. You will be meeting this constantly.]

2.5.1 Graphs of the Logarithms

The left-hand graph below is a reminder of the exponential dependence of y on $x = \log y$; the second below graph shows sketches of $\log_b y$ as a function of y for $b = 2$ and $b = 10$.



2.5.2 Using Logarithms

Since the log of a number is an exponent of some exponential, all we need to do to understand the properties of logarithms is to refer back to the properties of exponentials as summarized in Sections 2.1.1 and 2.1.2 of this review. These essential properties, which make logs so useful, are:

1) When two numbers in exponential form are multiplied together, the exponents add;

2) When one such number is divided by another, we subtract the exponents;

3) Since raising a number to a give power, p, means multiplying the number by itself p times, its exponent is multiplied by p;

Translated into the language of logs, these results become:

1′) To multiply two numbers together, we add their logs:

$$\log_b mn = \log_b m + \log_b n$$

2′) To divide one number by another, we subtract their logs:

$$\log_b \left(\frac{m}{n}\right) = \log_b m - \log_b n$$

3′) To raise a number to any power, we multiply its log by that power:

$$\log_b(n^p) = p \log_b n$$

Of course, what we need as an answer to any such calculation is not just the *log* of the product, quotient, or power, but the final numbers themselves. Therefore we have to go through the process of raising the base b to the power represented by the logarithm — i.e., by the left-hand sides of the above equations. Remember:

$$N = b^{\log_b N}.$$

This is the process of finding the so-called *antilogarithm* of those quantities. Once upon a time, this had to be done by referring to tables of such antilogs — just as the logs of the original numbers had to be read off from tables of logarithms. Nowadays, all we have to do is push the appropriate buttons on our pocket clculators (using the INVERSE operation to get antilogs — i.e., the final answers). But it's important to understand *in principle* what is involved. Below are some exercises to use these principles.

Exercise 2.5.2:
Given $\log_{10} 2 = 0.301$, and $\log_{10} 3 = 0.477$, find:

a) $\log_{10} 144$;

b) $\log_{10} \frac{8}{27}$;

c) $\log_{10}(2^{10})$.

Exercise 2.5.3:
Use your calculator to evaluate the *antilogs* (base 10) of the following logarithms:

a) 5;

b) 3.30103;

c) -0.69897;

d) the *sum* of (b) and (c);

e) the *difference* of (b) and (c).

(This means evaluating 10^x where x is the given logarithm.)

2.6 Calculating with Complex Numbers

A complex number is one of the form $a + bi$, where $i = \sqrt{-1}$ and a, b represent real numbers. The number a is called the *real part*, and b is called the *imaginary part* of the complex number $a + bi$.

Complex numbers are added and multiplied by treating them as polynomials in i; the only difference is that whenever i^2 occurs in the answer, it is replaced by -1. Thus:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

and

$$(a + bi)(c + di) = ac + (bc + ad)i + bdi^2 = (ac - bd) + (bc + ad)i$$

The *complex conjugate* of $a + bi$ is defined to be $a - bi$.

To divide two complex numbers, multiply top and bottom by the complex conjugate of the denominator. For example:

$$\frac{2+3i}{1-2i} = \frac{2+3i}{1-2i} \cdot \frac{1+2i}{1+2i} = \frac{-4+7i}{5} = -\frac{4}{5} + \frac{7}{5}i.$$

2.7 Answers to Exercises

Exercise 2.1.1: (a) 2^7; (b) 10^6; (c) 10^8; (d) 32/37; (e) $(1/8) \cdot (1/100) = 1/800$.

Exercise 2.1.2: (a) $x = 2$; (b) $x = 0$; (c) $x = 2^{-1/4}$; (d) $x = -10$

Exercise 2.1.3: (a) $2^4 = 16$; (b) $25^3 = 15625$; (c) 4/9; (d) $4^{3/2} = 8$; (e) $10^5 = 100,000$

Exercise 2.2.1: (a) $n = \log_{1.08} 1.25 = 3$; or by trial-and-error: if $1.25 = (1.08)^n$, try $n = 2$: $(1.08)^2 \cong 1.17 < 1.25$, $n = 3$: $(1.08)^3 \cong 1.25$, so $n = 3$ yrs.; (b) $223.16

Exercise 2.2.2: (a) 7^{3w}; (b) 15^{y+1}; (c) 2^8; (d) 2^{4a-b}

Exercise 2.4.1: (a) 7071; 80,000; (b) 2 days

Exercise 2.5.1: (a) 3; (b) 5; (c) -4; (d) 0; (e) 3; (f) doesn't exist, by log definition.

Exercise 2.5.2: (a) 2.158; (b) -0.528; (c) 3.01

Exercise 2.5.3: (a) 10,000; (b) 2,000; (c) 0.2; (d) 400; (e) 10,000

This module is based largely on an earlier module prepared by the MIT Mathematics Department.

3 Review Problems on Logarithms, Exponentials, and Complex Numbers

3.1 Calculating with exponents

(See Section 2.1 of the review module.)

Problem 1: Simplify each of the following; find the numerical value, if possible.

a) $(2^3)^{-2}(2^5)$ b) $(10^{-3})^4(2^3)(2^4)(5^9)$ c) $\frac{a^5b^{10}+a^3b^4}{(ab)^4}$
d) $8^{3/2}$ e) $16^{3/4}$ f) $27^{-4/3}$
g) $(.0016)^{-1/4}$ h) $(.064)^{2/3}$

Problem 2: Solve each of the following for x:

a) $81^x = 9 \cdot 3^x$ b) $10^{x^2} = (10^x)^2 \cdot 1000$ c) $\frac{2^x}{32} = (2^x)^4$
d) $2^{x^2} = 32\sqrt[3]{2}$ e) $4^{x+1} = (2^{2^3})(2^x)^3$ f) $27^{4/x} = 9 \cdot 3^{2/x}$

3.2 Calculating with logarithms

(See Section 2.5 of the review module.)

Note: we write $\log a = \log_{10} a$, $\ln a = \log_e a$ *(where* $e \approx 2.72$*). Logs to other bases are written explicitly as* $\log_b a$*.*

Problem 3: Simplify each of the following:

a) $\frac{\log 32}{\log 2}$ b) $\log_3(1/81)$ c) $\log(\sqrt[4]{100})^3$
d) $\ln(e^{kt})$ e) $10^{\log_{10} 2}$ f) $\ln 6 - \ln 3 + \ln \sqrt{2}$

Problem 4: Using the approximations $\log 2 \approx .3$, $\log 3 \approx .5$, $\ln 2 \approx .7$, $\ln 10 \approx 2.3$, calculate approximate values for each of the following: a) $\log 12$ b) $\ln 5$ c) $\log .75$ d) $\ln 16$ e) $\log 9/8$ f) $\ln 1/8$

Problem 5: Solve each of the following for x: (use the approximations given in problem 3.2)

a) $\log_b(x-2) = 0$ b) $\log x - \log(x-1) = 2$ c) $10^{2x} = 2^{10}$
d) $\ln x + \ln(x+1) = 1$ e) $e^{3x} = 8$

3.3 Problems involving exponentials and Logs

3.3.1 Calculating with Exponents

(See Section 2.4 of the module.)

Note: in these problems, use the approximations given in problem 3.2 above.

Problem 6: A colony of bacteria is growing according to the growth law

$$N = N_0 e^{3t}$$

where N is the number present at time t (days), and N_0 is the starting number. After how many days will the colony be four times as large? (Use $\ln 2 \approx .7$)

Problem 7: A colony of bacteria is growing according to the law $N = N_0 e^{kt}$, where N is the number at time t (hours), N_0 is the starting number, and k is a constant. The colony has doubled in size after 5 hours. Find the value of k.

Problem 8: The apparent brightness B of stars and planets is related to their magnitude m by the formula (B_0 is a constant):

$$B = B_0 \cdot 100^{-m/5}$$

Two stars, Krypton and Ryton, have respective magnitudes 4.0 and 1.5. What is the ratio (Krypton:Ryton) of their apparent brightness?

Problem 9: Referring to problem 3.3.1, give a formula for the magnitude of a star, in terms of its apparent brightness, and use it to answer this question: if Fyxx is 100 times brighter than Styx, by how much do their magnitudes differ?

Problem 10: A radioactive substance is decaying according to the law $A = A_0 e^{-\alpha t}$, where A is the amount at time t (years), A_0 is the starting amount, and α is a constant. If after 10 years one-quarter of the starting amount is left, find the value of α.

Problem 11: The acidity of a solution is measured by its pH, which is defined by:

$$\text{pH} = -\log[H],$$

where $[H]$ is the concentration of hydrogen ions in the solution. If acid #1 has a hydrogen ion concentration 30 times that of acid #2, what is the difference between their respective pH values ($\mathrm{pH}_1 - \mathrm{pH}_2$)?

Problem 12: The current i in a certain electrical circuit is falling according to the law

$$i = 40 \cdot e^{-3t},$$

where t is time in seconds. How long will it take for the current to fall from 40 to 5?

Problem 13: A heated object placed in an ice bath is cooling according to the law

$$\log T = \log T_0 - t/4,$$

where T is its temperature in degrees Celsius at time t (minutes), and T_0 is its starting temperature. If its starting temperature is 100, what will its temperature be (to the nearest degree) after 6 minutes?

Problem 14: When bank interest is compounded continuously, the amount A on deposit grows according to the formula

$$A = A_0 e^{rt},$$

where A_0 is the starting amount, A is the amount at time t (years), and r is the annual interest rate; assume it remains constant. If after 10 years the initial amount invest has doubled, what is the value of r?

Problem 15: The apparent loudness d of a sound (measured in decibels) is relative to its intensity I by the formula

$$d = 10 \log(I/I_0)$$

where I_0 is a constant (the intensity of a sound of 0 decibels). If a first sound is 15 decibels louder than a second sound, what is the ratio of their two intensities (first:second)? Give a numerical answer, with one significant figure.

Problem 16: A colony of bacteria grows exponentially, according to the law $A = A_0 e^{kt}$, where t is time (hours), and A is the amount present at time t. If it takes 35 hours for the colony to increase by a factor of 32, how long will it take the colony to increase by a factor of 10?

Problem 17: I get one dollar the first day, two dollars the second day, and each succeeding day I get twice what I got the day before. After about how many days will I have a million dollars? (Use the formula for the sum of a geometric progression – see Section 2.8 of the Algebra review module.)

3.4 Calculating with complex numbers

(See Section 2.6 of the review module.)

Problem 18: Calculate each of the following, expressing your answer in the form $a + bi$:

a) $(3-2i)(1+i)$ b) $(1+i)^3$ c) $(2+3i)^2(2-3i)^2$

d) $\frac{2+i}{1-i}$ e) $\frac{1+3i}{1-3i}$ f) $\frac{3+i}{i}$

Problem 19: If $(a+bi)(2-3i) = 1$, what are the real numbers a and b?

3.5 Solutions

Solution 1:

a) $(2^3)^{-2} \cdot 2^5 = 2^{-6} \cdot 2^5 = 2^{-1} = \frac{1}{2}$

b) $(10^{-3})^4 \cdot 2^3 \cdot 2^4 \cdot 5^9 = 10^{-12} \cdot 2^7 \cdot 5^9 = 10^{-12} \cdot 10^7 \cdot 5^2 = 10^{-5} \cdot 25 = 2.5 \cdot 10^{-4}$

c) $\frac{a^5b^{10}+a^3b^4}{(ab)^4} = ab^6 + \frac{1}{a}$

d) $8^{3/2} = (2^3)^{3/2} = 2^{9/2} = 2^4 \cdot 2^{1/2} = 16\sqrt{2}$

e) $16^{3/4} = (\sqrt[4]{16})^3 = 2^3 = 8$

f) $27^{4/3} = (\sqrt[3]{27})^4 = 3^4 = 81$, so $27^{-4/3} = 1/(27^{4/3}) = \frac{1}{81}$.

g) $(.0016)^{1/4} = (16 \times 10^{-4})^{1/4} = 16^{1/4} \cdot (10^{-4})^{1/4} = 2 \times 10^{-1} = .2$, so $(.0016)^{-1/4} = \frac{1}{.2} = 5$

h) $(.064)^{+2/3} = (64 \times 10^{-3})^{2/3} = 64^{2/3}(10^{-3})^{2/3} = 4^2(10^{-1})^2 = .16$

Solution 2:

a)

$$\begin{aligned} 81^x &= 9 \cdot 3^x \\ (3^4)^x &= 3^2 \cdot 3^x \\ 3^{4x} &= 3^{2+x} \\ 4x &= 2 + x \\ x &= 2/3 \end{aligned}$$

b)

$$\begin{aligned} 10^{x^2} &= (10^x)^2 \cdot 1000 \\ 10^{x^2} &= 10^{2x} \cdot 10^3 \\ x^2 &= 2x + 3 \\ x^2 - 2x - 3 &= 0 \\ (x-3)(x+1) &= 0x & &= 3, -1 \end{aligned}$$

c)

$$\begin{aligned} \frac{2^x}{32} &= (2^x)^4 \\ 2^x \cdot 2^{-5} &= 2^{4x} \\ x - 5 &= 4x \\ x &= -5/3 \end{aligned}$$

d)

$$\begin{aligned} 2^{x^2} &= 32\sqrt[3]{2} \\ 2^{x^2} &= 2^5 2^{1/3} \\ x^2 &= 5 + \frac{1}{3} = \frac{16}{3} \\ x &= \pm\frac{4}{3}\sqrt{3} = \pm\frac{4}{3}\sqrt{3} \end{aligned}$$

e)

$$
\begin{aligned}
4^{x+1} &= 2^{2^3} \cdot (2^x)^3 \\
2^{2(x+1)} &= 2^{2^3+3x} \\
2x+2 &= 8+3x \\
x &= -6
\end{aligned}
$$

f)

$$
\begin{aligned}
27^{4/x} &= 9 \cdot 3^{2/x} \\
(3^3)^{4/x} &= 3^2 \cdot 3^{2/x} \\
\frac{12}{x} &= 2 + \frac{2}{x} \\
\frac{10}{x} &= 2 \\
x &= 5
\end{aligned}
$$

Solution 3:

a) $\dfrac{\log 32}{\log 2} = \dfrac{\log 2^5}{\log 2} = \dfrac{5 \log 2}{\log 2} = 5$

b) $\log_3 \dfrac{1}{81} = \log_3 3^{-4} = -4 \log_3 = -4$

c) $\log(\sqrt[4]{100})^3 = \log 100^{3/4} = \dfrac{3}{4} \log 100 = \dfrac{3}{4} \cdot 2 = \dfrac{3}{2}$

d) $\ln e^{kt} = kt \ln e = kt$

e) $10^{\log 2} = 2$ by definition.

$$
\begin{aligned}
\text{or:} \qquad \text{let } 10^{\log 2} &= x \\
\text{then } \log 2 \log 10 &= \log x \\
\text{so } \log 2 &= \log x \\
2 &= x
\end{aligned}
$$

f) $\ln 6 - \ln 3 + \ln \sqrt{2} = \ln \frac{6}{3}\sqrt{2} = \ln 2\sqrt{2} = \ln 2 + \frac{1}{2}\ln 2 = \frac{3}{2}\ln 2$

Solution 4:

a) $\log 12 = \log 3 \cdot 2^2 = \log 3 + 2\log 2 \approx .5 + .6 = 1.1$

b) $\ln 5 = \ln \frac{10}{2} = \ln 10 - \ln 2 \approx 2.3 - .7 = 1.6$

c) $\log .75 = \log \frac{3}{2^2} = \log 3 - 2\log 2 \approx .5 - .6 = -.1$

d) $\ln 16 = \ln 2^4 = 4\ln 2 \approx 2.8$

e) $\log \frac{9}{8} = \log \frac{3^2}{2^3} = 2\log 3 - 3\log 2 \approx 1.0 - .9 = .1$

f) $\ln \frac{1}{8} = \ln 2^{-3} = -3\ln 2 \approx -2.1$

Solution 5:

a) $\log_b(x-2) = 0$, so $x - 2 = b^0 = 1$ and $x = 3$.

b)

$$\begin{aligned} \log x - \log(x-1) &= 2 \\ \log \frac{x}{x-1} &= 2 \\ \frac{x}{x-1} &= 10^2 = 100 \\ x &= 100x - 100 \\ x &= \frac{100}{99} \end{aligned}$$

c)

$$\begin{aligned} 10^{2x} &= 2^{10} \\ 2x \log 10 &= 10 \log 2 \\ 2x &\approx 3 \\ x &\approx \frac{3}{2} \end{aligned}$$

d)

$$\begin{aligned}
\ln x + \ln(x+1) &= 1 \\
\ln x(x+1) &= 1 \\
x(x+1) &= e \\
x^2 + x - e &= 0 \\
x &= \frac{-1+\sqrt{1+4e}}{2}
\end{aligned}$$

Note if $\ln x$ is defined, $x > 0$. We reject the negative solution.

e) $e^3 x = 8$ implies $3x = \ln 8 = \ln 2^3 = 3\ln 2$, so $x = \ln 2$.

Solution 6: We have $N = N_0 e^{3t}$, and want to know when N equals $4N_0$. We solve $4N_0 = N_0 e^{3t}$ for t:

$$\begin{aligned}
4N_0 &= N_0 e^{3t} \\
4 &= e^{3t} \\
2\ln 2 &= 3t \\
t &= \frac{2}{3}\ln 2 \approx \frac{2}{3}(.7) \approx .5
\end{aligned}$$

Solution 7:

$$\begin{aligned}
N &= N_0 e^{kt} \\
N &= 2N_0 \quad \text{when } t = 5 \\
\text{therefore, } 2N_0 &= N_0 e^{5k} \\
2 &= e^{5k} \\
\ln 2 &= 5k \\
k &= \frac{1}{5}\ln 2 \approx \frac{.7}{5} = .14
\end{aligned}$$

Solution 8:

$$
\begin{aligned}
B &= B_0 \cdot 100^{-m/5} \\
B_K &= B_0 \cdot 100^{-4.0/5} \text{ (Krypton)} \\
B_R &= B_0 \cdot 100^{-1.5/5} \text{ (Ryton)} \\
\frac{B_K}{B_R} &= 100^{\frac{-4.0+1.5}{5}} = 100^{-.5} \\
&= \frac{1}{100^{.5}} = \frac{1}{\sqrt{100}} = \frac{1}{10}
\end{aligned}
$$

Solution 9:

$$
\begin{aligned}
B &= B_0 \cdot 100^{-m/5} \\
\log B &= \log B_0 - \frac{m}{5} \log 100 \\
\log B &= \log B_0 - \frac{2}{5} m \\
m &= \boxed{\frac{5}{2} \left(\log B_0 - \log B \right)} \\
m_{\text{Fyxx}} - m_{\text{Styx}} &= \frac{5}{2} \left(- \log B_{\text{Fyxx}} + \log B_{\text{Styx}} \right)
\end{aligned}
$$

since $B_{\text{Fyxx}} = 100 B_{\text{Styx}}$, $\log B_{\text{Fyxx}} = 2 + \log B_{\text{Styx}}$

$$
\begin{aligned}
m_{\text{Fyxx}} - m_{\text{Styx}} &= \frac{5}{2} \left(-2 - \log B_{\text{Styx}} + \log B_{\text{Fyxx}} \right) \\
&= -5
\end{aligned}
$$

Solution 10:

$$
\begin{aligned}
A &= A_0 e^{-\alpha t} \\
&= \frac{1}{4} A_0 \quad \text{when } t = 10 \\
\frac{1}{4} A_0 &= A_0 e^{-10\alpha} \\
\ln \frac{1}{4} &= -10\alpha \\
-2\ln 2 &= -10\alpha \\
\alpha &= \frac{1}{5} \ln 2 \approx \frac{.7}{5} = .14
\end{aligned}
$$

Solution 11:

$$
\begin{aligned}
[H]_1 &= 30[H]_2 \\
\log[H]_1 &= \log 3 + \log 10 + \log[H]_2 \\
&\approx 1.5 + \log[H]_2 \\
\text{therefore,} \quad -\log[H]_1 &\approx -1.5 - \log[H]_2 \\
pH_1 &\approx -1.5 + pH_2 \\
pH_1 - pH_2 &\approx -1.5
\end{aligned}
$$

Solution 12:

$$
\begin{aligned}
i &= 40e^{-3t} \quad (i = 40 \text{ when } t = 0.) \\
\text{We want } t \text{ when } i &= 5 \\
5 &= 40e^{-3t} \\
\frac{1}{8} &= e^{-3t} \\
-3\ln 2 &= -3t \\
t = \ln 2 \approx .7 \text{ sec.}
\end{aligned}
$$

Solution 13:

$$\log T = \log T_0 - \frac{t}{4}$$
$$\text{when } t = 6 \text{ what is } T?$$
$$\log T = \log 100 - \frac{6}{4}$$
$$\log T = 2 - 3/2 = \frac{1}{2}$$
$$\text{therefore, } T = 10^{1/2} = \sqrt{10} \approx 3°C$$

Solution 14:

$$A = A_0 e^{rt}$$
$$A = 2A_0 \text{ when } t = 10\text{;What is } r?2A_0 \quad = A_0 e^{10r}$$
$$\ln 2 = 10r$$
$$.7 \approx 10r$$
$$r \approx .07$$
$$\boxed{7\%}$$

Solution 15:

$$d_1 = 10\log(I_1/I_0) \text{ (first sound)}$$
$$d_2 = 10\log(I_2/I_0) \text{ (second sound)}$$
$$d_1 - d_2 = 10\log\frac{I_1/I_0}{I_2/I_0}$$
$$= 10\log I_1/I_2$$
$$15 = 10\log I_1/I_2 \quad \text{(we're given } d_1 - d_2 = 15)$$
$$1.5 = \log I_1/I_2$$
$$\text{therefore, } I_1/I_2 = 10^{1.5} = 10\sqrt{10} \approx 30$$

Solution 16:

$$
\begin{aligned}
A &= A_0 e^{kt} \\
32A_0 &= A_0 e^{35k} \quad \text{(from given data)} \\
2^5 &= e^{35k} \\
5 \ln 2 &= 35k \\
k &= \boxed{\frac{1}{7} \ln 2}
\end{aligned}
$$

Continuing,

$$
\begin{aligned}
A_0 &= A_0 e^{kt} \\
10 &= e^{kt} \\
\ln 10 &= kt = \left(\frac{1}{7} \ln 2\right) t \\
t &= \frac{7 \ln 10}{\ln 2} \approx \frac{7 \cdot 2.3}{.7} \\
&\approx \boxed{23 \text{ hours}}
\end{aligned}
$$

Solution 17: After n days, I have:

$$\underset{\text{day 1}}{\$1} + \underset{\text{day 2}}{2} + \underset{\text{day 3}}{2^2} + 2^3 + \cdots + \underset{\text{day } n}{2^{n-1}}$$ which sums to:

$$\frac{2^n - 1}{2 - 1} \approx 2^n$$

We want to know when $2^n = 10^6$:

$$
\begin{aligned}
2^n &= 10^6 \\
n \log 2 &= 6 \\
n &\approx \frac{6}{.3} \approx 20 \text{ days.}
\end{aligned}
$$

Solution 18: Recall that: $(a + bi)(a - bi) = a^2 + b^2$.

a) $(3 - 2i)(1 + i) = 3 - 2i + 3i - 2i^2 = 3 + 2 + i = 5 + i$

b) $(1+i)^3 = 1 + 3i + 3i^2 + i^3 = 1 - 3 + 3i - i = -2 + 2i$

c) $(2+3i)^2(2-3i)^2 = [(2+3i)(2-3i)]^2 = [2^2+3^2]^2 = 169$

d) $\frac{2+i}{1-i} = \frac{2+i}{1-i} \cdot \frac{1+i}{1+i} = \frac{1+3i}{1^2+1^2} = \frac{1}{2} + \frac{3}{2}i$

e) $\frac{1+3i}{1-3i} = \frac{1+3i}{1-3i} \cdot \frac{1+3i}{1+3i} = \frac{-8+6i}{1^2+3^2} = -\frac{4}{5} + \frac{3}{5}i$

f) $\frac{3+i}{i} = \frac{3+i}{i} \cdot \frac{-i}{-i} = \frac{-3i+1}{1} = 1 - 3i$

Solution 19: The best way is to write $(a+bi)(2-3i) = 1$ in the form $a + bi = \frac{1}{2-3i}$. Then,

$$a + bi = \frac{1}{2-3i} \cdot \frac{2+3i}{2+3i} = \frac{2+3i}{2^2+3^2} = \frac{2}{13} + \frac{3}{13}i$$

You can also use "undetermined coefficents":

$$(a+bi)(2-3i) = (2a+3b) + (2b-3a)i = 1$$

so $2a + 3b = 1$ and $-3a + 2b = 0$. We can solve this system of linear equations by multiplying the first equation by 3 and the second by 2, then adding:

$$\begin{aligned} 6a + 9b &= 3 \\ -6a + 4b &= 0 \\ \\ 13b &= 3 \end{aligned}$$

Hence $b = \frac{3}{13}$ and $a = \frac{2}{3}b = \frac{2}{13}$.

$$\boxed{a + bi = \frac{2}{13} + \frac{3}{13}i}$$

4 Exponentials and Logarithms Self-Tests

4.1 Exponentials and Logarithms Diagnostic Exam #1

Problem 20: Evaluate $8^{-2/3}$.

Problem 21: Reduce and simplify $x^{\frac{3}{y-1}} x^{\frac{6}{y^2-1}}$

Problem 22: Solve for x: $\log_{10} x + \log_{10}(x+3) = 1$.
(One way is to begin by combining the logarithms.)

Problem 23: The apparent brightness B of stars is related to their magnitude m by:

$$B = c_0 10^{-m/5}, \qquad c_0 = \text{ constant.}$$

If the magnitudes of Ajax and Thorax are respectively -1.7 and 4.2, what is the ratio of their respective brightness?

Problem 24: Evaluate $\log_3 \frac{1}{81}$

Problem 25: The amplitude i of the current in a circuit is given by the formula:

$$i = 60e^{-rt},$$

where r is a constant, time t is measured in seconds, and e is the base for the natural logarithms. How long does it take for the current to decrease from 60 to 30? (Express your answer in terms of r and $\ln = \log_e$.)

4.2 Exponentials and Logarithms Diagnostic Exam #1 Solutions

Solution 20: Evaluate $8^{-2/3}$.

$8^{2/3} = (8^{1/3})^2 = (\sqrt[3]{8})^2 = 2^2 = 4$. Therefore, $8^{-2/3} = \frac{1}{8^{2/3}} = \boxed{\frac{1}{4}}$.

Solution 21: Reduce and simplify:

$$x^{\frac{3}{y-1}} x^{\frac{6}{y^2-1}}$$

We proceed by combining exponents:

$$x^{\frac{3}{y-1}+\frac{6}{y^2-1}} = x^{\frac{3(y+1)+6}{y^2-1}} = \boxed{x^{\frac{3y+9}{y^2-1}}}.$$

Solution 22: Solve for x: $\log_{10} x + \log_{10}(x+3) = 1$. (One way is to begin by combining the logarithms.)

$$\log_{10} x + \log_{10}(x+3) = \log_{10} x(x+3) = 1$$
$$\text{therefore, } \; x(x+3) = 10$$

Alternately,

$$x^2 + 3x - 10 = 0$$
$$(x+5)(x-2) = 0$$

Hence $x = -5$ or $x = 2$. We reject the solution $x = -5$, since $\log -5$ is undefined; thus the answer is $\boxed{x = 2}$.

Solution 23: The apparent brightness B of stars and planets is measured in terms of magnitude m, by the formula $B = c_0 10^{m/5}$, where c_0 is a constant. If the apparent magnitude of Venus is -4.2 and Jupiter's is -1.7, what is the ratio of their respective brightness?

$$\frac{B_{Ajax}}{B_{Thorax}} = \frac{c_0 10^{-1.7/5}}{c_0 10^{-4.2/5}} = 10^{(-1.7+4.2)/5} = 10^{2.5/5} = 10^{.5} = \sqrt{10}$$

Solution 24: Evaluate $\log_3 \frac{1}{81}$.

$$\log_3 \frac{1}{81} = \log_3 3^{-4} = -4 \log_3 3 = -4$$

Solution 25: The amplitude i of the current in a circuit is given by the formula:

$$i = 60e^{-rt},$$

where r is a constant, time t is measured in seconds, and e is the base for the natural logarithms. How long does it take for the current to decrease from 60 to 30? (Express your answer in terms of r and $\ln = \log_e$.)

$i = 60$ when $t = 0$; we want to find a value of t for which $i = 30$. We thus have:

$$\begin{aligned} 60e^{-rt} &= 30 \\ e^{-rt} &= 1/2 \\ -rt &= \ln 1/2 = -\ln 2 \\ \text{therefore} \quad t &= \boxed{\frac{\ln 2}{r}} \end{aligned}$$

4.3 Exponentials and Logarithms Diagnostic Exam #2

Problem 26: Evaluate $16^{-3/4}$.

Problem 27: If $(27)^{4/x} = 9 \cdot 3^{2/x}$, then what is x?

Problem 28: The apparent loudness d of a sound is is measured in decibels, and is related to the intensity I of the sound by the formula:

$$I = c_0 10^d, \qquad c_0 = \text{ constant.}$$

If an amplifier has a maxiumum sound of 20 decibels, how many such amplifiers will it take to produce a sound of 22 decibels? (Assume that doubling the number of amplifiers doubles the intensity of the maximum sound they produce.)

Problem 29: Evaluate $(\ln 32)/(\ln 2)$.

Problem 30: Superman decides to go for a two million mile run (approximately 10^{10} feet). On the first day, he runs one foot. The next day he runs two feet, and on eacy successive day he runs twice as far as he did the previous day. Approximately how many days does it take him to complete the run? (use $\log_{10} 2 \approx .3$)

(Recall the formula $1 + r + r^2 + r^3 + \cdots + r^{n+1} = \frac{r^n - 1}{r - 1}$.)

4.4 Logarithms and Exponentials Diagnostic Test #2 Solutions

Solution 26: Evaluate $16^{-3/4}$.

$16^{3/4} = (16^{1/4})^3 = (\sqrt[4]{16})^3 = 2^3 = 8$

Therefore, $16^{-3/4} = \frac{1}{16^{3/4}} = \boxed{\dfrac{1}{8}}$

Solution 27: If $(27)^{4/x} = 9 \cdot 3^{2/x}$, then what is x?

$$\begin{aligned}(27^{4/x})^x &= (9 \cdot 3^{2/x})^x \\ 27^4 &= 9^x \cdot 3^2 \\ \frac{(3^3)^4}{3^2} &= (3^2)^x \\ \frac{3^{12}}{3^2} &= 3^{2x} \\ 10 &= 2x \\ x &= \boxed{5}\end{aligned}$$

Solution 28: The apparent loudness d of a sound is is measured in decibels, and is related to the intensity I of the sound by the formula:

$$I = c_0 10^d, \qquad c_0 = \text{ constant.}$$

If an amplifier has a maxiumum sound of 20 decibels, how many such amplifiers will it take to produce a sound of 22 decibels? (Assume that doubling the number of amplifiers doubles the intensity of the maximum sound they produce.)

$I_{\text{amp}} = c_0 \cdot 10^{20}$

$I_{\text{new sound}} = c_0 \cdot 10^{22} = 10^2 I_{amp}$. Therefore, we need $\boxed{100}$ amplifiers.

Solution 29: Evaluate $(\ln 32)/(\ln 2)$.

$$\frac{\ln 32}{\ln 2} = \frac{\ln 2^5}{\ln 2} = \frac{5 \ln 2}{\ln 2} = \boxed{5}$$

Solution 30: Superman decides to go for a two million mile run (approximately 10^{10} feet). On the first day, he runs one foot. The next day he runs two feet, and on eacy successive day he runs twice as far as he did the previous day. Approximately how many days does it take him to complete the run? (use $\log_{10} 2 \approx .3$)

(Recall the formula $1 + r + r^2 + r^3 + \cdots + r^{n+1} = \frac{r^n - 1}{r-1}$.)

In n days he runs $1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1}$ feet, which equals $\frac{2^n - 1}{2-1} \approx 2^n$ feet.

We want 2^n to equal 10^{10}; we take $\log_{10}$ of both sides to get $n \log_{10} 2 = 10$. Thus, $n \approx 10/.3 \approx 33$ days.

5 Exponentials and Logarithms Self-Evaluation

You may want to informally evaluate your understanding of the various topic areas you have worked through in the *Self-Paced Review*. If you meet with tutors, you can show this evaluation to them and discuss whether you were accurate in your self-assessment.

For each topic which you have covered, grade yourself on a one to ten scale. One means you completely understand the topic and are able to solve all the problems without any hesitation. Ten means you could not solve any problems easily without review.

1.1	The Laws of Exponents	____
2.2	Fractional Exponents	____
2.	Exponentials as Functions	____
3.	Graphs of the Exponentials	____
4.	Logarithms	____
5.	Calculating with Logarithms	____
6.	Calculating with Complex Numbers	____