Contents

1 How to Use the Self-Paced Review Module

The *Self-Paced Review* consists of review modules with exercises; problems and solutions; self-tests and solutions; and self-evaluations for the four topic areas Algebra, Geometry and Analytic Geometry, Trigonometry, and Exponentials & Logarithms. In addition, previous *Diagnostic Exams* with solutions are included. Each topic area is independent of the others.

The *Review Modules* are designed to introduce the core material for each topic area. A numbering system facilitates easy tracking of subject material. For example, in Algebra, the subtopic Linear Equations is numbered with 2.3.Problems and self-evaluations are categorized using this numbering system.

When using the *Self-Paced Review*, it is important to differentiate between concept learning and problem solving. The review modules are oriented toward refreshing concept understanding while the problems and self-tests are designed to develop problems solving ability. When reviewing the modules, exercises are liberally sprinkled throughout the modules: solve these exercises when working through the module. The problems should be attempted without looking at the solutions. If a problem cannot be solved after at least two honest efforts, then consult the solutions. Trying many times and then succeeding results in a better understanding than trying several times and reading the solution.

The tests should be taken only when both understanding of the material and problem solving ability have been achieved. The self-evaluation is a useful tool to evaluate the mastery of the material. Finally, the previous Diagnostic Exams should provide the finishing touch.

The review modules were written by Professor AṖḞrench (Physics Department) and Adeliada Moranescu (MIT Class of 1994). The problems and solutions were written by Professor Arthur Mattuck (Mathematics Department). This document was originally produced by the Undergraduate Academic Affairs Office, August, 1992, and transcribed to LaTeX and edited for OCW by Tea Dorminy (MIT Class of 2013) in August, 2010.

2 Exponentials & Logarithms Review Module

Exponentials and logarithms could well have been included within the Algebra module, since they are basically just part of the business of dealing with powers of numbers or powers of algebraic quantities. But they have so much importance in their own right that it is convenient to give them a module of their own.

2.1 The Laws of Exponents

exp:a:i

The concept of exponent begins with the multiplication of a given quantity a with itself an arbitrary number of times:

$$a^m = \underbrace{a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdots}_{m \text{ times}}$$

On the left we have the product expressed in exponential notation, which is very compact and efficient. The expression above is in effect a definition of what we mean by an exponent.

If we multiply a by itself a total of $(m+n)$ times, we can of course write it as the m-fold product multiplied by the n-fold product:

$$\underbrace{a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdots}_{(m+n) \text{ times}} = (\underbrace{a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdots}_{m \text{ times}})(\underbrace{a \cdot a \cdot a \cdot a \cdot a \cdots}_{n \text{ times}})$$

This is far more easily expressed in exponential form, and gives us the first rule for dealing with quantities expressed in this way:

$$(a^m)(a^n) = a^{m+n}$$

When two quantities, written as powers of a given number, are multiplied together, we add the exponents.

Note that if we took the quantity a^m and multiplied it by itself p times, this would be the same as raising the quantity a^m to the pth power; or multiplying a by itself mp times. Therefore,

$$(a^m)^p = a^{mp}$$

.

When a quantity, written as a power of a given number, is itself raised to a certain power, the exponents multiply.

Be sure you keep clear in your mind the distinction between this formula and the previous one; the difference can be enormous if large powers are involved. Take, for instance, the following quantities:

$$(10^3)(10^6) = (10)^{3+6} = 10^9 = \text{ one billion}$$

$$(10^3)^6 = (10^3)(10^3)(10^3)(10^3)(10^3)(10^3) = 10^{18} = \text{ a billion billion!}$$

If we take the number a^m (the number a multiplied by itself m times) and *divide* it by a^n (a multiplied by itself n times), the result is a multiplied by itself $m - n$ times. Thus we have the rule for dividing one exponential by another:

$$\frac{a^m}{a^n} = a^{m-n}$$

We can see from this that the reciprocal of any positive power of a is an equal negative power:

$$\frac{1}{a^n} = a^{-n}$$

Also, if we put $m = n$ in the previous expression, the left-hand side is equal to 1. The right hand side is a^0. Thus, we have another result:

Any number (other than zero itself) raised to the power zero is equal to 1.

Exercise 2.1.0: `exp:two:one:one`

No calculators! Evaluate

1. $(2^4)(2^3)$
2. $(10^2)(10^4)$
3. $(10^2)^4$ (Compare with the previous)
4. $(2^5)(3^{-3})$
5. $(2^{-3})/(10^2)$

Exercise 2.1.1:
Solve for x:

exp:two:one:two

1. $x^5 = 32$
2. $2^x = 1$
3. $x^{-4} = 2$
4. $10^x = 0.0000000001$

The answers to the exercises are all collected together at the end of this module. We have tried to eliminate errors, but if you find anything that you think needs to be corrected, please write to us.

2.2 Fractional Exponents, Etc.

exp:a:ii

Go back to the expression for raising a quantity to a certain power and then reaising the resulting number to som eother power:

$$(a^m)^p = a^{mp}$$

If the product $mp = 1$, the right-hand side is just $a^1 = a$.

Suppose p is some specific integer; then what the above equation says is that the parenthetical expressoin, raised to the pth power, is equal to a. But this means that the parenthetical expression is what we define as the *pth root of a.* Also, since $mp = 1$, we must have $m = 1/p$. Thus:

A fractional power corresponds to taking roots of numbers:

$$a^{1/p} = \sqrt[p]{a}$$

We can proceed from this to consider a wider variety of exponentials:

Suppose we take the nth root of a and raise it to the power m. Expressed in exponential form, this can be written:

$$\left(\sqrt[n]{a}\right)^m = a^{(m/n)}$$

Thus we have a very convenient notation for writing any rational power of any root of a number.

This notation extends to negative powers too:

$$\frac{1}{\left(\sqrt[n]{a}\right)^m} = a^{-(m/n)}$$

Considering exponents as formed from products or ratios of integers — *rational numbers* — is enough for practical calculations, since these use only finite decimals, which are rational numbers. For example, $1.032 = 1032/1000$. For non-rational values of exponents, *limits* are used: $\sqrt{2} = 1.4142\cdots$, so $3^{\sqrt{2}}$ is the limit of 3^1, $3^{1.4}$, $3^{1.41}$, $3^{1.414}$, $3^{1.4142}$, etc.

Exercise 2.2.0: `exp:two:two:one`
Simplify and evaluate:

1. $\left(2^8\right)^{1/2}$
2. $\left(\sqrt{0.0016}\right)^{-3}$
3. $\left(\frac{8}{27}\right)^{\frac{2}{3}}$
4. $\left(4^{1/3}\right)^{\frac{9}{2}}$
5. $100^{5/2}$

Exponentials and Roots: Summary

$a^{m+n} = a^m a^n$	$a^{1/n} = \sqrt[n]{a}$ (n a positive integer)
$a^{-n} = 1/a^n$	$(a^m)^n = a^m n$
$a^0 = 1$	$a^m = a^n \to m = n$ (if $a \neq 1$)

2.3 Exponentials as Functions

Using limits, as above, we can ake the exponent to be any number we please, not just a rational number or fraction. In other words, we arrive at the concept that, in a quantity written as a^x, a can be any chosen number and x can be a continuous variable taking on any possible value from $-\infty$ to ∞. Thus a^x becomes a continuous function of x.

$$y(x) = a^x$$

This is a *exponential* function. A fixed (given) number a is raised to an arbitrary power x. The quantity a is the *base* of the exponential function. Contrast this with the function x^n: here a continuously variable number x is raised to a definite given power n.

Everyday life provides what is probably the most familiar example of an exponential function: the growth of a savings account with a fixed compound interest rate. If, for example, the interest is compounded annually, at a rate of 5%, then the amount A in the account after n years per dollar of initial deposit is given by

$$A(n) = (1.05)^n$$

However, in this computer age, interest may vary daily. If we again assume a 5% annual rate, the daily interst earned by \$1 is $0.05/365$, which is $0.00013698\cdots$. The interst is added to your initial dollar, yielding $1.00013698\cdots$ — almost insignificantly different from 1. In that case, after, say, 200 days, your initial dollar will be worth $(1.00013698\cdots)^2 00 =$ \$1.0277, a gain of almost 2.8 cents. (Check all of this on your calculator.) If you deposit \$600, the calculation above is done for each dollar in the deposit, so you end up with a total sum of $600(1.00013698\cdots)^{200} =$ \$616.66.

Exercise 2.3.0: `exp:two:three:one`
You put \$200 into a savings account. If the interest rate is 8% compounded annually, how many years will it take for your account to reach \$250? At the same annual rate, but compounded daily, what would be the balance in your account 500 days after the initial deposit?

The concept of the exponential function allows us to extend the range of quantities used as exponents. Besides being ordinary numbers ,they can be expressions involving variables that can be manupulated in the same way as numbers.

For example, $2^x 2^{-2x} = 2^{-x}$; $(10^{3x})^{1/x} = 10^3 = 1000$.

Equations with the unknown in the exponent can be solved: if $2^x = 4^{1/x}$, then $2^x = (2^2)^{1/x} = 2^{2/x}$, giving $x = 2/x$ and so $x = \pm\sqrt{2}$.

Watch the exponents! (This is an extension of what we said about integral exponents in exp:a:i 2.1.) It is important not to get confused when you see a compound exponent. The notation should make things clear. Consider the following two cases:

If you see $(10^x)^2$, you should read this as $(10^x)(10^x) = 10^{2x}$.

But if you see 10^{x^2}, you should read this as $10^{xx} = (10^x)^x$.

If, for example, you put $x = 5$, the first expression is equal to 10^{10}, which is a big number; but the second expression is equal to 10^{25}, which is 15 orders of magnitude bigger.

Exercise 2.3.1: exp:two:three:two

Combine and simplify:

1. $7^w 7^{2w}$
2. $(3 \cdot 5^y)(5 \cdot 3^y)$
3. $(2^4)^2$
4. $16^a/2^b$

2.4 Graphs of the Exponential Functions

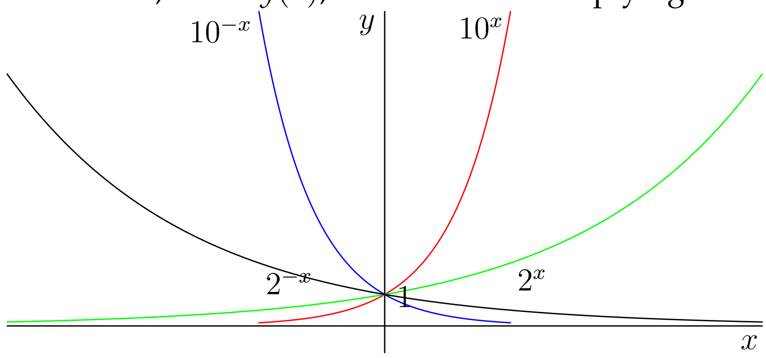
In mathematics and science, although the base a of the exponential funciton could in principle be any number, there are only three values of it that you will need to worry about for most purposes:

$a = 2$ This is the basis of binary algebra, as used in computer science, etc.

$a = 10$ This is the basis of many other scientific calculations.

$a = e$ The symbol e stands for a special irrational number whose first ten digits are 2.718281828. It is central to the use of exponential functions in calculus, but we will not consider it further here. However, if you have already studied some calculus, you will very likely have met it.

The graphs below show the general appearance of the exponential functions 2^x, 10^x, and their reciprocals $(1/2)^x = 2^{-x}$ and $(1/10)^x = 10^{-x}$. All exponential functions are equal to 1 at $x = 0$. To describe some exponential that has some specific value other than 1 at $x = 0$, we simply put this value, call it $y(0)$, in front as a multiplying or scaling factor.



Notice that with an exponential function the rate of change for a given change in x is independent of where you start – the initial value of x – it depends only on the difference between the initial and final values of x: if $y(x_1) = a^{x_1}$ and $y(x_2) = a^{x_2}$, then $\frac{y(x_1)}{y(x_2)} = a^{x_2 - x_1}$.

Exponentials show up in all sorts of contexts. Here are a few examples:

2.4.1 Positive Exponential Examples

Compound interest: as already discussed, $A(t) = A(0)(1+c)^t$, where c is the compound interest rate per unit of time and t is the time measured in those same units. For instance, if the rate is 5% compounded annually, then $c = 0.05$ and t is the time in years.

Growth of a biological population: a colony of bacteria, for example, grows by successive division, and may double in a few hours. Simplistically assuming the rate of reproduction is constant and that bacteria never die, one can put $N(n) = N(0)2^n$, where n is the number of doubling times (τ) since the population was equal to $N(0)$ — i.e., $n = t/\tau$.

2.4.2 Negative Exponential Examples

Radioactive decay: this, like biological growth, can be described in terms of the time to produce a factor 2 of change, but in this case a factor 2 decrease. This time is the *half-life*, $t_{1/2}$, and one has

$$N(t) = N(0)\left(\frac{1}{2}\right)^{\frac{t}{t_{1/2}}} = N(0)2^{-t/t_{1/2}}$$

Exercise 2.4.0: `exp:two:four:one`

1. A colony of bacteria in a test tube doubles every hour. If there are 2500 bacteria in the tube when the experimenter leaves for lunch at 12:00 noon, how many are there when she comes back at 1:30 PM? How many are there at 5:00 PM?

2. Find the half-life of a radioactive substance that has been left in a container for 6 days and decayed by a factor of 8.

2.5 Logarithms

If b is a fixed positive number other than 1, and if two other positive numbers x and y are related by $y = b^x$, we say that x is the *logarithm* of y to the base b. We write this relation in the form $x = \log_b y$.

This means that, for any (positive) number N and for any base b, the following relation always holds:

$$N = b^{\log_b N}$$

The word logarithm is too much a mouthful to use over and over again, so it is universally abbreviated as "log", not only in the above formula but also in speech.

Below are some exercises based on this definition:

Exercise 2.5.0: `exp:two:five:one`
Find:

1. $\log_{10} 1000$
2. $\log_2 32$
3. $\log_2(1/16)$
4. $\log_b 1$
5. $\log_3 27$
6. $\log_2(-4)$

The last set of exercise illustrates that:

- Logarithms of numbers greater than 1 are positive
- Logarithms of numbers between 0 and 1 exclusve are negative
- The logarithm of 1 is zero in any base
- You can't take a logarithm of a negative number — at least, until you get to complex numbers.

This last result follows from the definition of a logarithm. If $y = b^x$, with b a positive number, then y is positive; its smallest possible value is 0, when $x = -\infty$. In other words, $\log_b 0 = -\infty$, regardless of the value of the base b.

2.5.1 Logarithms to Base e

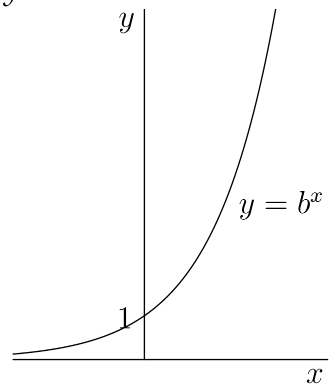
Although we are not using the number e as the base of logarithms here, we should draw attention to the fact that the symbol for such logarithms is written as ln often (for "natural logarithm") without any explicit definition of the base:

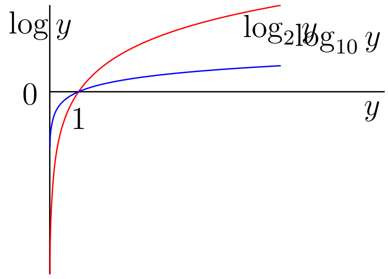
$$\ln x = \log_e x$$

It is universally understood that any logarithm written in this way is to base e. You will be meeting this constantly. Further, in some of these contexts, log written without a base is considered to be $\log_{10}$; in other contexts where ln is not used, log occasionally means ln when used without a base. The conventions of log without a base are highly confusing, and it is recommended that you always write explicit bases.

2.6 Graphs of Logarithms

The first below graph is a reminder of the exponential dependence of y on $x = \log y$; the second below graph shows sketchs of $\log_b y$ as a function of y for $b = 2$ and $b = 10$.





2.7 Using Logarithms

Since the log of a number is an exponent of some exponential, all we need to do to understand the properties of logarithms is to refer back to the properties of exponentials as summarized in Sections 2.1 and 2.2 of this review. These essential properties, which make logs so useful, are:

- When two numbers in exponential form are multiplied together, the exponents add;

- when one such number is divided by another, we subtract the exponents;
- Since raising a number to a give power p is equivalent to multiplying the number by itself p times, its exponent is multiplied by p;

Translated into the language of logs, these results become:

- To multiply two numbers together, we add their logs:

$$\log_b mn = \log_b m + \log_b n$$

- To divide one number by another, we subtract their logs:

$$\log_b \left(\frac{m}{n}\right) = \log_b m - \log_b n$$

- To raise a number to any power, we multiply its log by that power:

$$\log_b (n^p) = p \log_b n$$

Of course, what we need as an answer to any such calculation is not just the log of the poroduct, quotient, or power, but the final numbers themselves. Therefore we have to go through the porcess of raising the base b to the power represented by the logarithm — i.e.. by the left-hand sides of the above equations. Remember:

$$N = b^{\log_b N}$$

This is the process of finding the so-called *antilogarithm* of those quantities. Once upon a time this had to be done by referring to tables of such antilogs – just as the logs of the original numbers had to be read off from tables of logarithms. Nowadays, all we have to do is push the appropriate buttons on our pocket clculators, using the Inverse operation to get antilogs, the final answers. But it's important to understand in principle what is involved. Below are some exercises to use these principles.

Exercise 2.7.0:
Given $\log_{10} 2 = 0.301$, and $\log_{10} 3 = 0.477$, find $\log_{10} 144$, $\log_{10} \frac{8}{27}$, and $\log_{10}(2^{10})$.

`exp:two:seven:one`

Exercise 2.7.1: `exp:two:seven:two`
Use your calculator to evaluate the antilogs base 10 of the following logarithms:

1. 5
2. 3.30103
3. -0.69897
4. the sum of b and c
5. the difference of b and c

(This means evaluating 10^x where x is the given logarithm.)

3 Answers to Exercises

Exercise `exp:two:one:one` 2.1.1: (a) 2^7; (b) 10^6; (c) 10^8; (d) $32/37$; (e) $(1/8) \cdot (1/100) = 1/800$.

Exercise `exp:two:one:two` 2.1.2: (a) $x = 2$; (b) $x = 0$; (c) $x = 2^{-1/4}$; (d) $x = -10$

Exercise `exp:two:two:one` 2.2.1: (a) $2^4 = 16$; (b) $25^3 = 15625$; (c) $4/9$; (d) $4^{3/2} = 8$; (e) $10^5 = 100000$

Exercise `exp:two:three:one` 2.3.1: (a) $n = \log_{1.08} 1.25 = 3$; or by trial-and-error: if $1.25 = (1.08)^n$, try $n = 2$ and $n = 3$.; (b) \$223.16

Exercise `exp:two:three:two` 2.3.2: (a) 7^{3w}; (b) 15^{y+1}; (c) 2^8; (d) 2^{4a-b}

Exercise `exp:two:four:one` 2.4.1: (a) 7071; 80000; (b) 2 days

Exercise `exp:two:five:one` 2.5.1: (a) 3; (b) 5; (c) -4; (d) 0; (e) 3; (f) doesn't exist, by log definition.

Exercise `exp:two:seven:one` 2.7.1: (a) 2.158; (b) -0.528; (c) 3.01

Exercise `exp:two:seven:two` 2.7.2: (a) 10000; (b) 2000; (c) 0.2; (d) 400; (e) 10000

This module is based largely on an earlier module prepared by the MIT Mathematics Department.

4 Review Problems on Logarithms, Exponentials, and Complex Numbers

4.1 Calculating with exponents

Problem 0: Simplify each of the following; find the numerical value, if possible without a calculator.

1. $(2^3)^{-2}(2^5)$
2. $(10^{-3})^4(2^3)(2^4)(5^9)$
3. $\frac{a^5b^{10}+a^3b^4}{(ab)^4}$
4. $8^{3/2}$
5. $16^{3/4}$
6. $27^{-4/3}$
7. $(.0016)^{-1/4}$
8. $(.064)^{2/3}$

Problem 1: Solve each of the following for x:

1. $81^x = 9 \cdot 3^x$
2. $10^{x^2} = (10^x)^2] \cdot 1000$
3. $\frac{2^x}{32} = (2^x)^4$
4. $2^{x^2} = 32\sqrt[3]{2}$
5. $4^{x+1} = (2^{2^3})(2^x)^3$
6. $27^{4/x} = 9 \cdot 3^{2/x}$

4.2 Calculating with logarithms

Problem 2: Simplify each of the following:

1. $\frac{\log 32}{\log 2}$ (use any base you please)
2. $\log_3(1/81)$
3. $\log(\sqrt[4]{100})^3$
4. $\ln e^{kt}$
5. $10^{\log_{10} 2}$
6. $\ln 6 - \ln 3 + \ln \sqrt{2}$

Problem 3: Using the approximations $\log 2 \approx .3$, $\log 3 \approx .5$, $\ln 2 \approx .7$, $\ln 10 \approx 2.3$, calculate approximate values for each of the following:

1. $\log 12$
2. $\ln 5$
3. $\log .75$
4. $\ln 16$
5. $\log 9/8$
6. $\ln 1/8$

Problem 4: Solve each of the following for x, using the approximations above:

1. $\log_b(x-2) = 0$
2. $\log x - \log(x-1) = 2$

3. $10^{2x} = 2^{10}$
4. $\ln x + \ln(x+1) = 1$
5. $e^{3x} = 8$

4.3 Problems involving exponentials and logs

Note: use the approximations above.

Problem 5: A colony of bacteria is growing according to the growht law

$$N = N_0 e^{3t}$$

where N is the number present at time t in days, and N_0 is the starting number. After how many days will the colony be four times as large?

Problem 6: A colony of bacteria is growing according to the law $N = N_0 e^{kt}$ where N is the number at time t in hours, N_0 is the starting number, and k is a constant. The colony has doubled in size after 5 hours. Find the value of k.

Problem 7: The apparent brightness B of stars and planets is related to their magnitude m by the formula (B_0 is a constant):

$$B = B_0 \cdot 100^{-m/5}$$

Two stars, Krypton and Ryton, have respective magnitudes 4.0 and 1.5. What is the ratio (Krypton:Ryton) of their apparent brightness?

Problem 8: In the previous problem involving Krypton and Ryton, give a formula for the magnitude of a star in terms of its apparent brightness, and use it to answer this question: if Fyxx is 100 times brighter than Styx, by how much do their magnitudes differ?

Problem 9: A radioactive substance is decaying according to the law $A = A_0 e^{-\alpha t}$, where A is the amount at time t (years), A_0 is the starting amount, and α is a constant. If after 10 years one-quarter of the starting amount is left, find the value of α.

Problem 10: The acidity of a solution is measured by its pH, which is defined by

$$pH = -\log[H]$$

where $[H]$ is the concentration of hydrogen ions in the solution. If acid #1 has a hydrogen ion concentration 30 times that of acid #2, what is the difference between their pH values?

Problem 11: The current i in a certain electrical current is falling according to the law

$$i = 40 \cdot e^{-3t}$$

where t is the time in seconds. How long will it take for the current to fall from 40 to 5?

Problem 12: A heated object placed in an ice bath is cooling according to the law

$$\log T = \log T_0 - t/4$$

where T is its temperature in degrees Celsius at time t (minutes), and T_0 is its starting temperature. If its starting temperature is 100, what will its temperature be, to the nearest degree, after 6 minutes?

Problem 13: When bank interest is compounded continuously, the amount A on deposit grows according to the formula

$$A = A_0 e^{rt}$$

where A_0 is the starting amount, A is the amount at time t (years), and r is the annual interest rate; assume it stays constant. If after 10 years the initial amount invest has doubled, what is the value of r?

Problem 14: The apparent loudness d of a sound (measured in decibels) is relative to its intensity I by the formula

$$d = 10\log(I/I_0)$$

where I_0 is a constant, the intensity of a sound of 0 decibels. If a loud rock concert is 15 decibels louder than a motorcycle, what is the ratio of their two intensities (concert:motorcycle)? Give a numerical answer with one significant figure.

Problem 15: A colony of bacteria grows exponentially, according to the law $A = A_0 e^{kt}$, where t is time (hours) and A is the amount present at time t. If it takes 35 hours for the colony to increase by a factor of 32, how long will it take the colony to increase by a factor of 100?

Problem 16: I get one dollar the first day, two dollars the second day, and each succeeding day I get twice what I got the day before. After about how many days will I have a million dollars? Use the formula for the sum of a geometric progression, as in the Algebra review problems.

4.4 Introduction to Complex Numbers

A complex number is one of the form $a + bi$, where i is defined as $\sqrt{-1}$ and a, b represent real numbers. The number a is called the *real part*, and b is called the *imaginary part* of the complex number $a + bi$.

Complex numbers are added and multiplied by treating them as polynomials in i; the only difference is that $i^2 = -1$. Thus:

$$(a+bi)+(c+di) = (a+c)+(b+d)i$$

and

$$(a+bi)(c+di) = ac + (bc+ad)i + bdi^2 = (ac-bd)+(bc+ad)i$$

The *complex conjugate* of $a+bi$ is defined to be $a-bi$, since $(a+bi)(a-bi) = a^2 - b^2i^2 = a^2+b^2$.

To divide two complex numbers, multiply top and bottom by the complex conjugate of the denominator. For example,

$$\frac{2+3i}{1-2i} = \frac{2+3i}{1-2i}\cdot\frac{1+2i}{1+2i} = \frac{-4+7i}{5} = -\frac{4}{5}+\frac{7}{5}i$$

4.5 Problems involving complex numbers

Problem 17: Calculate each of the following, expressing your answer in the form $a+bi$:

1. $(3-2i)(1+i)$
2. $(1+i)^3$
3. $(2+3i)^2(2-3i)^2$
4. $\frac{2+i}{1-i}$
5. $\frac{1+3i}{1-3i}$
6. $\frac{3+i}{i}$

Problem 18: If $A(2-3i) = 1$, for some complex number A, what is A? Express in the form $a+bi$.

5 Solutions to Logarithm, Exponent, and Complex Number Review Problems

Solution to problem 0:

1. $(2^3)^{-2} \cdot 2^5 = 2^{-6} \cdot 2^5 = 2^{-1} = \frac{1}{2}$
2. $(10^{-3})^4 \cdot 2^3 \cdot 2^4 \cdot 5^9 = 10^{-12} \cdot 2^7 \cdot 5^9 = 10^{-12} \cdot 10^7 \cdot 5^2 = 10^{-5} \cdot 25 = 2.5 \cdot 10^{-4}$
3. $\frac{a^5 b^{10} + a^3 b^4}{(ab)^4} = ab^6 + \frac{1}{a}$
4. $8^{3/2} = (2^3)^{3/2} = 2^{9/2} = 2^4 \cdot 2^{1/2} = 16\sqrt{2}$
5. $16^{3/4} = (\sqrt[4]{16})^3 = 2^3 = 8$
6. $27^{4/3} = (\sqrt[3]{27})^4 = 3^4 = 81$ so $27^{-4/3} = 1/(27^{4/3}) = \frac{1}{81}$.
7. $(.0016)^(.25) = (16 \times 10^{-4})^{1/4} = 16^{1/4} \cdot (10^{-4})^{1/4} = 2 \times 10^{-1} = .2$ so $(.0016)^{-1/4} = \frac{1}{.2} = 5$
8. $(.064)^(2/3) = (64 \times 10^{-3})^{2/3} = 64^{2/3}(10^{-3})^{2/3} = 4^2(10^{-1})^2 = .16$

Solution to problem 1:

1.

$$\begin{aligned} 81^x &= 9 \cdot 3^x \\ (3^4)^x &= 3^2 \cdot 3^x \\ 3^{4x} &= 3^{2+x} \\ 4x &= 2 + x \\ x &= 2/3 \end{aligned}$$

2.

$$\begin{aligned} 10^{x^2} &= (10^x)^2 \cdot 1000 \\ 10^{x^2} &= 10^2 x \cdot 10^3 \\ x^2 &= 2x + 3 \\ x^2 - 2x - 3 &= 0 \\ (x-3)(x+1) &= 0 \end{aligned}$$

$x = 3, -1$

3.

$$
\begin{aligned}
\frac{2^x}{32} &= (2^x)^4 \\
2^x \cdot 2^{-5} &= 2^{4x} \\
x - 5 &= 4x \\
x &= -5/3
\end{aligned}
$$

4.

$$
\begin{aligned}
2^{x^2} &= 32\sqrt[3]{2} \\
2^{x^2} &= 2^5 2^{1/3} \\
x^2 &= \frac{16}{3} \\
x &= \pm\frac{4}{3}\sqrt{3}
\end{aligned}
$$

5.

$$
\begin{aligned}
4^{x+1} &= 2^{2^3} \cdot (2^x)^3 \\
2^{2(x+1)} &= 2^{2^3+3x} \\
2x + 2 &= 8 + 3x \\
x &= -6
\end{aligned}
$$

6.

$$
\begin{aligned}
27^{4/x} &= 9 \cdot 3^{2/x} \\
(3^3)^{4/x} &= 3^2 \cdot 3^{2/x} \\
\frac{12}{x} &= 2 + \frac{2}{x} \\
\frac{10}{x} &= 2 \\
x &= 5
\end{aligned}
$$

Solution to problem 2:

1.
$$\frac{\log 32}{\log 2} = \frac{\log 2^5}{\log 2} = \frac{5\log 2}{\log 2} = 5$$

2.
$$\log_3 \frac{1}{81} = \log_3 3^{-4} = -4\log_3 = -4$$

3.
$$\log(\sqrt[4]{100})^3 = \log 100^{3/4} = \frac{3}{4}\log 100 = \frac{3}{4}\cdot 10 = \frac{3}{2}$$

(Note the usage of $\log = \log_{10}$ in these problems.)

4. $\ln e^{kt} = kt \ln e = kt$

5. $10^{\log 2} = 2$ by definition.

6.
$$\ln 6 - \ln 3 + \ln\sqrt{2} = \ln\frac{6}{3}\sqrt{2} = \ln 2\sqrt{2} = \ln 2 + \frac{1}{2}\ln 2 = \frac{3}{2}\ln 2$$

Solution to problem 3:

1. $\log 12 = \log 3\cdot 2^2 = \log 3 + 2\log 2 \approx .5 + .6 = 1.1$

2. $\ln 5 = \ln 10/2 = \ln 10 - \ln 2 \approx 2.3 - .7 = 1.6$

3. $\log .75 = \log\frac{3}{2^2} = \log 3 - 2\log 2 \approx .5 - .6 = -.1$

4. $\ln 16 = \ln 2^4 = 4\ln 2 \approx 2.8$

5. $\log 9/8 = \log 3^2/2^3 = 2\log 3 - 3\log 2 \approx 1.0 - .9 = .1$

6. $\ln\frac{1}{8} = \ln 2^{-3} = -3\ln 2 \approx -2.1$

Solution to problem 4:

1. $\log_b(x-2) = 0 \rightarrow x - 2 = b^0 = 1 \rightarrow x = 3$

2.

$$\begin{aligned} \log x - \log(x-1) &= 2 \\ \log \frac{x}{x-1} &= 2 \\ \frac{x}{x-1} &= 10^2 = 100 \\ x &= 100x - 100 \\ x &= \frac{100}{99} \end{aligned}$$

3.

$$\begin{aligned} 10^{2x} &= 2^{10} \\ 2x \log 10 &= 10 \log 2 \\ 2x &\approx 3 \\ x &\approx \frac{3}{2} \end{aligned}$$

4.

$$\begin{aligned} \ln x + \ln(x+1) &= 1 \\ \ln x(x+1) &= 1 \\ x(x+1) &= e \\ x^2 + x - e &= 0 \\ x &= \frac{-1+\sqrt{1+4e}}{2} \end{aligned}$$

Note that $x > 0$, as $\ln x$ is defined; so we reject the negative solution for x.

5. $e^3 x = 8 \to 3x = \ln 8 = \ln 2^3 = 3\ln 2$ so $x = \ln 2$.

Solution to problem 5: We have $N = N_0 e^{3t}$, and want to know when is $N = 4N_0$: we solve $4N_0 = N_0 e^{3t}$ for t.

$$\begin{aligned} 4 &= e^{3t} \\ 2\ln 2 &= 3t \\ t &= \frac{2}{3}\ln 2 \approx \frac{2}{3}(.7) \approx .5 \end{aligned}$$

Solution to problem 6:

$$\begin{aligned}
N &= N_0 e^{kt} \\
N &= 2N_0, t = 5 \\
2N_0 &= N_0 e^{5k} \\
2 &= e^{5k} \\
\ln 2 &= 5k \\
k &= \frac{1}{5}\ln 2 \approx \frac{.7}{5} \approx .14
\end{aligned}$$

Solution to problem 7:

$$\begin{aligned}
B &= B_0 \cdot 100^{-m/5} \\
B_K &= B_0 \cdot 100^{-4/5} \text{ (Krypton)} \\
B_R &= B_0 \cdot 100^{-1.5/5} \text{ (Ryton)} \\
\frac{B_K}{B_R} &= 100^{\frac{-4.0+1.5}{5}} = 100^{-.5} \\
&= \frac{1}{100^{.5}} = \frac{1}{10}
\end{aligned}$$

Solution to problem 8:

$$\begin{aligned}
B &= B_0 \cdot 100^{-m/5} \\
\log B &= \log B_0 - \frac{m}{5}\log 100 \\
\log B &= \log B_0 - \frac{2}{5}m \\
m &= \boxed{\frac{5}{2}(\log B_0 - \log B)} \\
m_{Fyxx} - m_{Styx} &= \frac{5}{2}\left(-\log B_{Fyxx} + \log B_{Styx}\right) \\
(\text{since } B_F = 100 B_S) &= \frac{5}{2}(-2 - \log B_S + \log B_F) \\
&= -5
\end{aligned}$$

Solution to problem 9:

$$\begin{aligned} A &= A_0 e^{-\alpha t} \\ &= \frac{1}{4} A_0, t = 10 \\ \frac{1}{4} A_0 &= A_0 e^{-10\alpha} \\ \ln \frac{1}{4} &= -10\alpha \\ -2 \ln 2 &= -10\alpha \\ \alpha &= \frac{1}{5} \ln 2 \approx .14 \end{aligned}$$

Solution to problem 10:

$$\begin{aligned} [H]_1 &= 30[H]_2 \\ \log[H]_1 &= \log 3 + \log 10 + \log[H]_2 \\ &= 1.5 + \log[H]_2 \\ -\log[H]_1 &= -1.5 - \log[H]_2 \\ pH_1 &= -1.5 + pH_2 \\ pH_1 - pH_2 &= -1.5 \end{aligned}$$

Solution to problem 11:

$$\begin{aligned} i &= 40e^{-3t} \\ i &= 40, t = 0 \\ &\text{Want } i = 5 \\ 5 &= 40e^{-3t} \\ \frac{1}{8} &= e^{-3t} \\ -3 \ln 2 &= -3t \\ t = \ln 2 \approx .7 \end{aligned}$$

Solution to problem 12:

$$\begin{aligned}
\log T &= \log T_0 - \frac{t}{4} \\
t = 6 &\rightarrow T = ? \\
\log T &= \log 100 - \frac{6}{4} \\
\log T &= 2 - 3/2 = \frac{1}{2} \\
T &= 10^{1/2} = \sqrt{10} \approx 3^\circ C
\end{aligned}$$

Solution to problem 13:

$$\begin{aligned}
A &= A_0 e^{rt} \\
A &= 2A_0, t = 10 \\
2A_0 &= A_0 e^{10r} \\
\ln 2 &= 10r \\
.7 &\approx 10r \\
r &\approx .07 \\
&\boxed{7\%}
\end{aligned}$$

Solution to problem 14:

$$\begin{aligned}
d_1 &= 10\log(I_1/I_0) \text{ (concert)} \\
d_2 &= 10\log(I_2/I_0) \text{ (motorcycle)} \\
d_1 - d_2 &= 10\log\frac{I_1/I_0}{I_2/I_0} \\
&= 10\log I_1/I_2 \\
15 &= 10\log I_1/I_2 \\
1.5 &= \log I_1/I_2 \\
I_1/I_2 &= 10^{1.5} = 10\sqrt{10} \approx 30
\end{aligned}$$

Solution to problem 15:

$$\begin{aligned}
A &= A_0 e^{kt} \\
32A_0 &= A_0 e^{35k} \\
2^5 &= e^{35k} \\
5\ln 2 &= 35k \\
k &= \boxed{\frac{1}{7}\ln 2}
\end{aligned}$$

$$\begin{aligned}
10A_0 &= A_0 e^{kt} \\
10 &= e^{kt} \\
\ln 10 &= kt = \left(\frac{1}{7}\ln 2\right) t \\
t &= \frac{7\ln 10}{\ln 2} \approx \frac{2.3}{.7}(.7) \\
&\approx \boxed{23}
\end{aligned}$$

Solution to problem 16: After n days, I have

$$1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1}$$

which sums to

$$\frac{2^n - 1}{2 - 1} \approx 2^n$$

We want to know when $2^n = 10^6 \rightarrow n \log 2 = 6 \rightarrow n \approx \frac{6}{.3} \approx 20$ days.

Solution to problem 17:

1. $(3-2i)(1+i) = 3 - 2i + 3i - 2i^2 = 3 + 2 + i = 5 + i$
2. $(1+i)^3 = 1 + 3i + 3i^2 + i^3 = 1 - 3 + 3i - i = -2 + 2i$
3. $(2+3i)^2(2-3i)^2 = ((2+3i)(2-3i))^2 = (2^2+3^2)^2 = 169$
4. $\frac{2+i}{1-i} = \frac{2+i}{1-i} \cdot \frac{1+i}{1+i} = \frac{1+3i}{1^2+1^2} = \frac{1}{2} + \frac{3}{2}i$
5. $\frac{1+3i}{1-3i} = \frac{1+3i}{1-3i} \cdot \frac{1+3i}{1+3i} = \frac{-8+6i}{1^2+3^2} = -\frac{4}{5} + \frac{3}{5}i$

6. $\frac{3+i}{i} = \frac{3+i}{i} \cdot \frac{-i}{-i} = \frac{-3i+1}{1} = 1 - 3i$

Solution to problem 18: The best way is to write $A(2-3i) = 1$ in the form $A = \frac{1}{2-3i}$. Then,

$$\frac{1}{2-3i} \cdot \frac{2+3i}{2+3i} = \frac{2+3i}{2^2+3^2} = \frac{2}{13} + \frac{3}{13}i$$

You can also solve, since you know if $A = a + bi$ a and b are real:

$$(a+bi)(2-3i) = (2a+3b) + (2b-3a)i = 1$$

so $2a + 3b = 1$ and $2b - 3a = 0$; solve these, and we obtain the same value of A.

6 Exponentials and Logarithms Diagnostic Test #1

Problem 19: Evaluate $8^{-2/3}$.

Problem 20: Reduce and simplify

$$x^{\frac{3}{\tau-1}} x^{\frac{6}{\tau^2-1}}$$

Problem 21: Solve for x: $\log_{10} x + \log_{10}(x+3) = 1$.

Problem 22: The apparent brightness B of stars and planets is measured in terms of magnitude m, by the formula $B = c_0 10^{m/5}$, where c_0 is a constant. If the apparent magnitude of Venus is -4.2 and Jupiter's is -1.7, what is the ratio of their respective brightness?

Problem 23: The apparent brightness B of stars is related to their magnitude m by

$$B = c_0 10^{-m/5}$$

with c_0 a constant. If the magnitudes of Ajax and Thorax are 1.7 and 4.2 respectively, what is the ratio of their respective brightness?

Problem 24: Evaluate $\log_3 \frac{1}{81}$

Problem 25: The amplitude of the current in a certain electrical circuit decays exponentially. After time $t = 0$, the amplitude is given by $60(10^{-\alpha t})$. Using the approximation that $\log_{10} 2 \approx .3$, how long does it take for the current's amplitude to decay by a factor of one half?

7 Solutions to Exponentials and Logarithms Diagnostic Test #1

Problem: Evaluate $8^{-2/3}$.

Solution to problem 19: $8^{2/3} = (8^{1/3})^2 = (\sqrt[3]{8})^2 = 2^2 = 4$; since $a^{-b} = \frac{1}{a^b}$, $8^{-2/3} = \frac{1}{8^{2/3}} = \boxed{\frac{1}{4}}$.

Problem: Reduce and simplify

$$x^{\frac{3}{\tau-1}} x^{\frac{6}{\tau^2-1}}$$

Solution to problem 20: We begin by combining exponents:

$$x^{\frac{3}{\tau-1}+\frac{6}{\tau^2-1}}$$

And set a common denominator in the exponent of $\tau^2 - 1$:

$$x^{\frac{3(\tau+1)+6}{\tau^2-1}} = \boxed{x^{\frac{3\tau+9}{\tau^2-1}}}$$

Problem: Solve for x: $\log_{10} x + \log_{10}(x+3) = 1$.

Solution to problem 21:

$$\begin{aligned} \log_{10} x + \log_{10}(x+3) = \log_{10} x(x+3) &= 1 \\ x(x+3) &= 10 \\ x^2 + 3x - 10 &= 0 \\ (x+5)(x-2) &= 0 \end{aligned}$$

We discard $x = -5$, since $\log -5$ is undefined; thus $x = \boxed{2}$.

Problem: The apparent brightness B of stars and planets is measured in terms of magnitude m, by the formula $B = c_0 10^{m/5}$, where c_0 is a constant. If the apparent magnitude of Venus is -4.2 and Jupiter's is -1.7, what is the ratio of their respective brightness?

Solution to problem 22:

$$\frac{B_{Venus}}{B_{Jupiter}} = \frac{c_0 10^{4.2/5}}{c_0 10^{1.7/5}} = 10^{(4.2-1.7)/5} = 10^{.5} = \sqrt{10}$$

Problem: The apparent brightness B of stars is related to their magnitude m by

$$B = c_0 10^{-m/5}$$

with c_0 a constant. If the magnitudes of Ajax and Thorax are 1.7 and 4.2 respectively, what is the ratio of their respective brightness?

Solution to problem 23:

$$\frac{B_{Ajax}}{B_{Thorar}} = \frac{c_0 10^{4.2/5}}{c_0 10^{1.7/5}} = 10^{(4.2-1.7)/5} = 10^{.5} = \sqrt{10}$$

Problem: Evaluate $\log_3 \frac{1}{81}$

Solution to problem 24:

$$\log_3 \frac{1}{81} = \log_3 3^{-4} = -4 \log_3 3 = -4$$

Problem: The amplitude of the current in a certain electrical circuit decays exponentially. After time $t = 0$, the amplitude is given by $60(10^{-\alpha t})$. Using the approximation that $\log_{10} 2 \approx .3$, how long does it take for the current's amplitude to decay by a factor of one half?

Solution to problem 25: $i = 60$ when $t = 0$; we want to find for what t $i = 30$. We thus have:

$$\begin{aligned} 60e^{-rt} &= 30 \\ e^{-rt} &= 1/2 \\ -rt &= \ln .5 = -\ln 2 \\ t &= \boxed{\frac{\ln 2}{r}} \end{aligned}$$

8 Exponentials and Logarithms Diagnostic Test #2

Problem 26: Evaluate $16^{-3/4}$.

Problem 27: If $(27)^{4/x} = 9(3)^{2/x}$, what is x?

Problem 28: The apparent loudness d of a sound is related to its intensity I by $I = c_0 10^d$ where c_0 is a constant. If the apparent loudness of an amplifier is 60 decibels and a jet plane is 120 decibels, then how many simultaneous amplifiers would it take to have the same intensity of sound as a jet plane? Assume intensity is additive: three amplifiers have three times the intensity of one amplifier.

Problem 29: Evaluate $(\ln 32)/(\ln 2)$.

Problem 30: Superman decides to go for a two million mile run, approximately 10^{10} feet. On the first day, Superman runs one foot. The next day he runs two feet, and on successive days he runs twice as far as on the previous day. How many days does it take for Superman to finish? (Hint: $2^{10} = 1024 \approx 1000 = 10^3$.)

9 Solutions to Logarithms and Exponentials Diagnostic Test #2

Problem: Evaluate $16^{-3/4}$.

Solution to problem 26: $16^{3/4} = (16^{1/4})^3 = (\sqrt[4]{16})^3 = 2^3 = 8$
$16^{-3/4} = \frac{1}{16^{3/4}} = \boxed{\frac{1}{8}}$
Problem: If $(27)^{4/x} = 9(3)^{2/x}$, what is x?

Solution to problem 27:

$$\begin{aligned} 27^x &= 9 \cdot 3^{2x} \\ 3^{3x} &= 3^2 \cdot 3^{2x} = 3^{2+2x} \\ 3x &= 2 + 2x \\ x &= \boxed{2} \end{aligned}$$

Problem: The apparent loudness d of a sound is related to its intensity I by $I = c_0 10^d$ where c_0 is a constant. If the apparent loudness of an amplifier is 60 decibels and a jet plane is 120 decibels, then how many simultaneous amplifiers would it take to have the same intensity of sound as a jet plane?

Solution to problem 28: $I_{amp} = c_0 \cdot 10^{20}$ and $I_{jet} = c_0 \cdot 10^{22} = 10^2 I_{amp}$, so we need $\boxed{100}$ amplifiers.

Problem: Evaluate $(\ln 32)/(\ln 2)$.

Solution to problem 29:

$$\frac{\ln 32}{\ln 2} = \frac{\ln 2^5}{\ln 2} = \frac{5 \ln 2}{\ln 2} = 5$$

Problem: Superman decides to go for a two million mile run, approximately 10^{10} feet. On the first day, Superman runs one foot. The next day he runs two feet, and on successive days he runs twice as far as on the previous day. How many days does it take for Superman to finish? (Hint: $2^{10} = 1024 \approx 1000 = 10^3$.)

Solution to problem 30: In n days he runs $1 + 2 + 4 + \cdots + 2^{n-1}$ feet, or $\frac{2^n - 1}{2-1}$ feet, which is approximately (1 less than) 2^n feet.

We would like 2^n to be about 10^{10}; we take $\log_{10}$ of both sides, to get $n \log_{10} 2 = 10 \to n \approx 10/.3 \approx 33$ days.