Self-Paced Study Guide in Algebra

June 29, 2010

1 How to Use This Guide

The *Self-Paced Review* consists of review modules with exercises; problems and solutions; self-tests and solutions; and self-evaluations for the four topic areas Algebra, Geometry and Analytic Geometry, Trigonometry, and Exponentials & Logarithms. In addition, previous *Diagnostic Exams* with solutions are included. Each topic area is independent of the others.

The *Review Modules* are designed to introduce the core material for each topic area. A numbering system facilitates easy tracking of subject material. For example, in Algebra, the subtopic Linear Equations is numbered with **??**. Problems and self-evaluations are categorized using this numbering system.

When using the *Self-Paced Review*, it is important to differentiate between concept learning and problem solving. The review modules are oriented toward refreshing concept understanding while the problems and self-tests are designed to develop problems solving ability. When reviewing the modules, exercises are liberally sprinkled throughout the modules: solve these exercises when working through the module. The problems should be attempted without looking at the solutions. If a problem cannot be solved after at least two honest efforts, then consult the solutions. Trying many times and then succeeding results in a better understanding than trying several times and reading the solution.

The tests should be taken only when both understanding of the material and problem solving ability have been achieved. The self-evaluation is a useful tool to evaluate the mastery of the material. Finally, the previous Diagnostic Exams should provide the finishing touch.

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2 Algebra Review

One widely used algebra textbook[1] begins with the sentence: "Algebra is really arithmetic in disguise." Arithmetic makes use of specific numbers; algebra develops general results that can be applied regardless of what the particular numbers are. THat makes algebra much more powerful than arithmetic. But if you want to *apply* a given algebraic result, you have to replace the xs and ys by specific numbers, and you have to know how to handle numerical quantities. So this algebra review begins with some stuff about numbers as such.

2.1 Scientific Notation

Writing numbers in what is called 'scientific notation' is an absolute necessity in science and engineering: you will be using it all the time. It is based on the fact that any positive number, however large or however small, can be written as a number between 1 and 10 multiplied by a power of 10. (And, for negative numbers, we simply put a $-$ sign in front.)

Begin by recalling that:

$$\begin{aligned} 10^1 &= 10 \\ 10^2 &= 10 \cdot 10 = 100 \\ 10^3 &= 10 \cdot 10 \cdot 10 = 1000 \\ &\text{etc.} \end{aligned}$$

To *multiply* together any two powers of 10, we simply *add* the exponents:

$$(10^m)(10^n) = 10^{(m+n)}$$

These and related matters are discussed in more detail in the module on Exponentials & Logarithms.

To *divide* one power of 10 by another, we *subtract* the exponents:

$$\frac{10^m}{10^n} = 10^{m-n}.$$

Thus $10^9/10^4 = 10^(9-4) = 10^5 = 100000$.

[1] Hughes-Hallett, *The Math Workshop: Algebra*, WẆṄorton Co., New York, 1980.

It is implicit in this that the reciprocal of a positive power of 10 is an equal negative power:

$$\frac{1}{10^n} = 10^{-n}.$$

The powers-of-10 notation also defines what is meant by 10^0:

$$10^0 = \frac{10^n}{10^n} = 1.$$

This is all very simple, but you need to be careful when negative powers of 10 are involved. Give yourself some practice by doing the following exercises:

Exercise 2.1.0:
Write the following numbers in scientific notation:

1. 80,516
2. 0.0751
3. 3,520,000
4. 0.000000081

Note: The answers to the exercises are all collected together at the end of this module. We have tried to eliminate errors, but if you find anything that you think needs to be corrected, please write to us.

A Note about Notation: We are using the center dot, for instance, $1.5 \cdot 10^4$, when we write a number as the product of a decimal with a power of 10. Perhaps you are accustomed to using the multiplication sign, $\times$, for this purpose, and you wlil find that both conventions are widely used. We prefer to use the center dot (especially in algebra and, later, in calculus) or, in some places, a parenthesis, to avoid any possible confusion with the variable x.

Exercise 2.1.1:
Evaluate:

1. $10^9 \cdot 10^{-3}$
2. $10^9 / 10^{-4}$
3. $\frac{10^{-19}}{10^{-34}}$

Answers? You don't need *us* to provide them! Just get the practice.

Now consider multiplying or dividing two numbers that are *not* pure powers of 10.

Take, for example, $2.6 \cdot 10^3$ and $5.3 \cdot 10^4$.

Their *product* is given by:

$$(2.6 \cdot 10^3)(5.3 \cdot 10^4) = (2.6)(5.3)(10^3 \cdot 10^4) = 13.8 \cdot 10^{3+4} = 13.8 \cdot 10^7 = 1.38 \times 10^8.$$

Their *quotient* is given by:

$$\frac{2.6 \cdot 10^3}{5.3 \cdot 10^4} = \left(\frac{2.6}{5.3}\right) 10^{3-4} = 0.68 \cdot 10^{-1} = 6.8 \cdot 10^{-2}.$$

Notice how we don't stop until we have converted the answer into a number between 1 and 10 multiplied by a power of 10.

Exercise 2.1.2:
Evaluate, in scientific notation:

1. $(1.5 \cdot 10^4)(7.5 \cdot 10^{-5})$
2. $(4.3 \cdot 10^{-6}) / (3.1 \cdot 10^{-10}$
3. $\frac{(1.2 \cdot 10^{-5})(1.5 \cdot 10^3)}{9 \times 10^{-9}}$

To *add* or *subtract* numbers in scientific notation, you first have to rearrange the numbers so that all the powers of 10 are the same; then you can add or subtract the decimal parts, leaving the power of 10 alone. It's usually best if you first express all the numbers in terms of the *highest* power of 10. In this connection, remember that, with negative powers of 10, smaller exponents means bigger numbers: 10^{-3} is bigger than 10^{-5}. You may, however, need to make a final adjustment if the combination of the decimal numbers is more than 10 or less than 1.

Examples:

$$\begin{aligned}
2.1 \cdot 10^3 + 3.5 \cdot 10^5 &= (0.021 + 3.5) \cdot 10^5 & &= 3.521 \cdot 10^5 \\
1.3 \cdot 10^8 - 8.4 \cdot 10^7 &= (1.3 - 0.84) \cdot 10^8 & &= 0.46 \cdot 10^8 = 4.6 \cdot 10^7 \\
9.5 \cdot 10^{-11} + 9.8 \cdot 10^{-12} &= (9.5 + 0.98) \cdot 10^{-11} & &= 10.48 \cdot 10^{-11} = 1.048 \cdot 10^{-10}
\end{aligned}$$

Exercise 2.1.3:
Evaluate:

1. $9.76 \cdot 10^9 + 7.5 \cdot 10^8$
2. $1.25 \cdot 10^6 - 7.85 \cdot 10^5$
3. $4.21 \cdot 10^{25} - 1.85 \cdot 10^{26}$
4. $4.05 \cdot 10^{-19} - 10^{-20}$
5. $(1.2 \cdot 10^{-19})(5.2 \cdot 10^{10} + 4 \cdot 10^9)$

2.2 Significant Figures

In the above examples and exercises in adding or subtracting numbers expressed in scinetific notation, you will have noticed that you may end up with a large number of digits. But not all of these may be significant. For example, when we added $2.1 \cdot 10^3$ and $3.5 \cdot 10^5$, we got $3.521 \cdot 10^5$.

However, each of the numbers being combined were given with only two-digit accuracy. (We are assuming that 3.5 means simply that the number is closer to 3.5 than it is to 3.4 or 3.6. If it meant 3.500 then these extra zero digits should have been included.) This means that the $2.1 \cdot 10^3$ did not add anything significant to the bigger number, and we were not justified in giving more than two digits in the final answer, which therefore should have been given as just $3.5 \cdot 10^5$.

Note, however, that if we were asked to add, say, $8.6 \cdot 10^3$ to $3.5 \cdot 10^5$, the smaller number would make a significant contribution to the final answer. The straight addition would give us $3.586 \cdot 10^5$. Rounding this off to two digits would then give the answer $3.6 \cdot 10^5$.

The general rules governing significant figures are:

1. The final answer should not contain more digits than are justified by the least accurate of the numbers being combined; **but**

2. Accuracy contained in the numbers being combined should not be sacrificed in the rounding-off process.

A few examples will help spell out these conditions. (We'll ignore the powers-of-10 factors for this purpose.)

Addition:	$1.63 + 2.1789 + 0.96432 = 4.77422$, rounded to 4.77.
Subtraction:	$113.2 - 1.43 = 111.77$, rounded to 111.8.
Multiplication:	$(11.3)(0.43) = 4.859$, rounded to 4.9 (only two digits justified).
Division:	$\frac{1.30}{0.43} = 3.02325814$, rounded to 3.0 (two digits).

Beware of your calculator! In the last examples, we used a pocket calculator to do the divisions. THis automatically gave 10 digits in each answer. *Most* of these digits are insignificant! Whenever you use your calculator for such a purpose, always ask yourself how many digits are justified and should be retained in the answer. Most of the calculations you will be doing will probably involve numbers with only a few significant digits. Get into the habit of cutting down your final answers to the proper size. As you can see, this situation will arise most importantly when you are doing divisions with your calculator. But watch out for suplus digits in multiplications as well.

Now let's turn to algebra proper.

2.3 Linear Equations

2.3.1 Equations in one variable

These scarcely need any discussion. There is one unknown, say x, and an equation that relates this to given numbers or constants. The only job is to tidy things up so as to solve for x explicitly.

Exercise 2.3.0:
Solve for x:

1. $5\left(x+\frac{1}{4}\right)=2x-\frac{1}{8}$
2. $\frac{3}{x}-\frac{4}{5}=\frac{1}{x}+\frac{1}{3}$
3. $3(ax+b)=5bx+c$

2.3.2 Simultaneous Equations in Two Variables

However two-variable equations are originally written, they can always be reduced to the following form:

$$\begin{aligned}a_1x+b_1y&=c_1\\ a_2x+b_2y&=c_2\end{aligned}$$

where the coefficients $a_1, a_2, b_1, b_2, c_1, c_2$ may be positive or negative.

Two essentially equivalent methods can be used to obtain the solutions for x and y:

1. *Substitution:* Use one of the equations (say the first) to get one of the unknowns (say y) in terms of x and known quantities. Substitute into the other equation to solve for the remaining unknown (x). Plug this value of x back into either of the initial equations to get y.

2. *Elimination:* Multiply the original equations by factors that make the coefficient of one unknown (say y) the same in both. By subtraction, eliminate y; this leads at once to x:

Multiply 1st eq. by b_2:

$$a_1 b_2 x + b_1 b_2 y = b_2 c_1$$

Multiply 2nd eq. by b_1:

$$a_2 b_1 x + b_2 b_1 y = b_1 c_2$$

Subtract:

$$(a_1 b_2 - a_2 b_1) x = b_2 c_1 - b_1 c_2$$

Thus:

$$x = \frac{b_2 c_1 - b_1 c_2}{a_1 b_2 - a_2 b_1}.$$

Then solve for y as before. But the last equation above points to a special situation: If the denominator $a_1 b_2 - a_2 b_1$ is 0, then x becomes indeterminate, and therefore so does y.

Keep it neat! Good housekeeping in mathematics is very important. It will make it easier for you to check your work and it will help you to avoid errors. Notice how we put the equals signs underneath one another in the above analysis. Make a practice of doing this yourself. Concentrate on making this a habit, and don't throw it out the window when you are taking a quiz or exam. You're bound to benefit from being neat and orderly.

Exercise 2.3.1:
Solve for both unknowns:

1.

$$2x - 3y = 4$$
$$3x - 2y = 5$$

2.

$$5a = b - 6$$
$$2b = a + 4$$

3.

$$\frac{3}{x} + \frac{4}{y} = 5$$
$$\frac{4}{x} - \frac{2}{y} = 3$$

[In 3, resist the temptation to multiply both equations throughout by xy to clear the denominators. Just put $1/x = u$, $1/y = v$, and solve first for (u, v), which are just as legitimate variables as (x, y).]

2.4 Polynomials

Much of what you do in algebra (and later, in calculus) will; be based on a familiarity with expressions made up of a sum of terms like $10x^3$ or $(3.2y^5)$ — in other words, sums of products of numbers called coefficients and powers of variables such as x. Such an expression is called a *polynomial* — meaning simply something with many terms. Many important polynomials are made up of a set of terms each of which contains a different power of a single quantity x. We can then write

$$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n,$$

where the quantities $a_0, a_1, a_2, a_3, \cdots, a_n$ are constant coefficients, labeled here to show which power of x they are associated with. We have written the combination as $P(x)$, to indicate that this combination, P, is a certain *function* of x if x is a mathematical variable. That is an aspect of polynomials which is important in calculus, but we won't expand on it in these algebra review notes.

A polynomial has a highest power of x in it; this is called the *degree* of the polynomial. Thus if the highest term is $3x^4$, we say that the polynomial is "of degree four" or "a quartic." For the most part, we shall not go beyond quadratics, of degree 2.

Many polynomials are a binomial (two-term) combination of two variables, of the form $(x+y)$, raised to an arbitrary power. The simplest examples of this type are $(x+y)$ itself and $(x+y)^2 = x^2 + 2xy + y^2$. A *binomial expansion* is the result of multiplying out such an expression, like $(2x+y)(x+y)(x+3y)$, into a polynomial.

If we want to add one polynomial to another, we simply add the terms with the same power of x.

Multiplication of two polynomials is a bit more complicated, but again involves identifying all the terms that have the same power of x and adding the coefficients to make a single term of the form $a_n x^n$. For instance:

$$\begin{aligned}
(x+2)(2x^2-3x+4) &= x(2x^2-3x+4) + 2(2x^2-3x+4) \\
&= (2x^3-3x^2+4x) + (4x^2-6x+8) \\
&= 2x^3 + (-3x^2+4x^2) + (4x-6x) + 8 \\
&= 2x^3 + x^2 - 2x + 8
\end{aligned}$$

Exercise 2.4.0:
Multiply $3x^2 + 4x - 5$ by $2x - 1$.

2.5 Quadratic Equations

A quadratic equation is a polynomial equality that can be manipulated into the form

$$ax^2 + bx + c = 0.$$

We can solve the equation through the process of *completing the square*. This technique is the basis of the general quadratic formula that you are likely familiar with already:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We'll give a specific example and then derive the general formula.

2.5.1 Completing the Square: An Example

Suppose we are asked to solve the equation

$$x^2 - 6x - 4 = 0$$

We recognize that the combination $x^2 - 6x$ is the first two terms of $(x-3)^2$. We just evaluate the complete square and see what is left over:

$$\begin{aligned}(x-3)^2 &= x^2 - 6x + 9 \\ x^2 - 6x &= (x-3)^2 - 9 \\ x^2 - 6x - 4 &= \Big((x-3)^2 - 9\Big) - 4 \\ &= (x-3)^2 - 13\end{aligned}$$

But $x^2 - 6x - 4 = 0 \rightarrow (x-3)^2 - 13 = 0$, requiring $(x-3) = \pm\sqrt{13}$. Thus the solution is

$$x = 3 \pm \sqrt{13}$$

Note that, in this equation, our original quadratic expression reduces to a perfect square *minus* a certain number, 13. When we set the whole expression equal to zero, this means that the perfect square is equal to a *positive* numnber, and we can proceed to take the square root of both sides. But if the quadratic had been a perfect square *plus* some number n we would have arrived at an equation of the form

$$(x+p)^2 = -n$$

We should then have been faced with taking the square root of a negative number. That would take us into the realm of *imaginary* numbers, which is beyond the scope of anything you need consider for the present.

2.5.2 The General Quadratic Formula

It is possible to obtain the quadratic formula directly from the equation

$$ax^2 + bx + c = 0$$

Subtract the constant c from both sides, to obtain

$$ax^2 + bx = -c$$

Divide both sides by the coefficient a:

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Complete the square of the left-hand side by adding $(b/2a)^2$. Add the same quantity to the right side:

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(x + \frac{b}{2a}\right)^2 = -fracca + \left(\frac{b}{2a}\right)^2$$

Bring the right-hand side to a common denominator, and take the square root of both sides:

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

Subtract $\frac{b}{2a}$ from both sides, to obtain an expression for x:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Now we have the general quadratic formula.

Notice that, if the solutions are to be real numbers, we must have $b^2 - 4ac \geq 0$.

Exercise 2.5.0:
Solve these quadratic equations by completing the square. If the coefficient of x^2 is not 1, you should follow the steps listed above in the derivation of the general formula.

1. $x^2 - 4x - 12 = 0$
2. $9x^2 - 6x - 1 = 0$
3. $x + 2x^2 = \frac{5}{8}$
4. $(x + 2)(2x - 1) + 3(x + 1) = 4$

Exercise 2.5.1:
Solve these quadratics with the general formula:

1. $2x^2 = 9x - 8$
2. $y = ut - \frac{1}{2}gt^2$ (solving for t)
3. $5x(x + 2) = 2(1 - x)$
4. $0.2x^2 - 1.5x = 3$
5. $x^2 - 2sx = 1 - 2s^2$ (solving for x)

2.6 Long Division & Factoring

Dividing one polynomial by another is very much like long division in arithmetic. We shall assume that the polynomial we are dividing into (the *dividend*) is of a higher degree than the one we are dividing by (the *divisor*). The reason for doing the division may be to find out if the higher-degree polynomial is exactly divisible by the lower-degree one; if it is, we have identified a way of *factoring* the higher-degree polynomial. We write both dividend and divisor in decreasing powers of x, or whatever variable is being used.

For example, we divide $4x^2 + 8x + 3$ by $2x + 1$:

$$\begin{array}{rr}
 & 2x + 3 \\ \cline{2-2}
2x+1 \big) & 4x^2 + 8x + 3 \\
 & -\,4x^2 - 2x \phantom{{}+3} \\ \cline{2-2}
 & 6x + 3 \\
 & -\,6x - 3 \\ \cline{2-2}
 & 0
\end{array}$$

It works! $2x + 1$ divides exactly into $4x^2 + 8x + 3$; the result is $2x + 3$, and so we have factored the quadratic into two linear factors.

> **Exercise 2.6.0:**
> Divide $5x^2 - 6x - 8$ by $x - 2$.

Let's now change the above example of long division slightly, by making our dividend $4x^2 + 8x + 5$. There will now be a remainder of 2.

The remainder 2 means that the quotient still has a fraction in it.

$$\frac{4x^2 + 8x + 5}{2x + 1} = 2x + 3 + \frac{2}{2x + 1}$$

This is analogous to the similar process with numbers; for instance,

$$\frac{423}{10} = 42 + \frac{3}{10}$$

With numbers we know that the remainder is always a number less than the divisor when dividing by 10; the possible remainders are $0, 1, \cdots, 9$. For polynomails, the remainder is always of lower degree than the divisor. For instance,

$$\frac{a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0}{x^2 + 1} = Q(x) + \frac{R(x)}{x^2 + 1}$$

The quotient $Q(x)$ has degree $3 = 5 - 2$ and the remainder $R(x)$ has degree at most 1. In other words, there are constants $b_0, b_1, b_2, b_3, c_0, c_1$ for which

$$Q(x) = b_3x^3 + b_2x^2 + b_1x + b_0 \text{ and } R(x) = c_1x + c_0$$

Exercise 2.6.1:
Divide $2x^3 + 2x^2 - 10x + 4$ by $x + 3$. What is the degree of the remainder?

Exercise 2.6.2:
If the polynomial $x^5 + 4x^4 - 6x^3 + 5x^2 - 2x + 3$ were divided by $x^2 + 2$, what would be the degree of the quotient and the degree of the remainder? Don't actually do the division!

Exercise 2.6.3:
What happens if you divide a polynomial of lower degree by one of higher degree?

Factoring is an important part of many calculations. But it is often-hard to see which polynomial evenly (exactly) divides another. In fact, factoring polynomials and numbers is an important problem for both mathematicians and computer scientists. Here is the only tool you need to know for now:

If $P(r) = 0$, then $(x - r)$ divides $P(x)$ evenly.

This is what we mean by saying that r is a *root* of $P(x)$.

Example: $P(x) = x^2 + x - 2$.

$P(x)$ has the root -2, since $P(-2) = (-2)^2 + (-2) - 2 = 0$. Then $P(x)$ is exactly divisible by $(x - (-2)) = x + 2$:

$$\begin{array}{r|l} & \quad\; x - 1 \\ \hline x+2 & x^2 \;\; + x - 2 \\ & -x^2 - 2x \\ \hline & \qquad -x - 2 \\ & \qquad\;\; x + 2 \\ \hline & \qquad\quad\; 0 \end{array}$$

The quotient is $x - 1$. Therefore, $x^2 + x - 2 = (x + 2)(x - 1)$.

The actual process of division here is of course just like that in the example at the beginning of the section. But there we gave the value of the divisor, instead of looking for it from scratch.

When dividing by a first order factor that divides evenly, you may save time by solving for coefficients using the following scheme:

$$\begin{aligned} x^2 + x - 2 &= (x + 2)(ax + b) \\ &= ax^2 + (2a + b)x + 2b \end{aligned}$$

Equating coefficients gives $a = 1, b = -1$. (But you can see in this case that $a = 1$ without writing down anything; then you can solve for b in $x^2 + x - 2 = (x + 2)(x + b)$.)

Here is another approach to factoring: For polynomials $x^n + a_{n-1}x^{n-1} + \cdots + a_0$ with leading coefficient 1, and integers for the other coefficients, any integer root must be a divisor of the constant term a_0. Thus in the polynomial $x^2 + x - 2$ we have $a_0 = -2$, and there are four integers that might work: ± 1 and ± 2.

Finally, for quadratic equations the roots are always obtainable by the quadratic formula:

$$x^2 + x - 2 = 0 \to x = \frac{-1 \pm \sqrt{1^2 - 4(-2)}}{2} = \frac{-1 \pm 3}{2} = -2 \text{ or } 1$$

In general, $ax^2 + bx + c = a(x - r_1)(x - r_2)$ where r_1 and r_2 are the roots. If the quadratic formula leads to the square root of a negative number than the quadratic has no real roots and no real factors. If you allow the use of complex numbers as coefficients, then the quadratic factors as usual.

Look out for easily factorable quadratics! Be on the alert for the following:

- Quadratics whose coefficients are small or simple enough for you to make a shrewd guess at the factorization
- Quadratics that are perfect squares
- Expressions of the form $a^2x^2 - b^2$, which can be immediately factored into $(ax+b)(ax-b)$

Don't overuse the quadratic formula when factoring a polynomial like $x^2 + x - 2$. Your first instinct should be to check ± 1 and ± 2 as roots. If you do use the quadratic formula, you should be prepared to *doublecheck* your answer. Arithmetic errors can be as significant in a physical problem as the difference between going the right way and the wrong way down a one-way street.

Exercise 2.6.4:
Test whether the following quadratics can be factored, and find the factors if they exist:

1. $8x^2 + 14x - 15$
2. $2x^2 - 3x + 10$
3. $9x^2 - 24ax + 16a^2$
4. $2x^2 + 10x - 56$
5. $100x^4 - 10^{10}$ (This is of 4th degree, but first put $x^2 = u$.)

Exercise 2.6.5:
Solve these quadratic equations by factoring:

1. $x^2 - 5x + 6 = 0$
2. $2x^2 - 5x - 12 = 0$
3. $3x^2 - 5x = 0$
4. $6x^2 - 7x - 20 = 0$

2.7 Some Tricks of the Trade

2.7.1 Getting Rid of Radicals

(Mathematical, not political)

Radicals are a nuisance if one is trying to solve an algebraic equation, and one usually wants to get rid of them. The way to get rid of a radical is of course to square it (or cube for a cube root, etc.); butr this only works if the radical stands by itself on one side of an equation. Thus if you have $\sqrt{x-1} + a = b$, you don't gain anything by squaring both sides. That simply gives you $x - 1 + 2a\sqrt{x-1} + a^2 = b^2$.

But, if you first isolate the radical on the left-hand side, you have

$$\sqrt{x-1} = b - a$$

Then when you square, you get $x - 1 = (b-a)^2$, and you are in business.

You may have to go through this routine more than once.

Example: Suppose you have $\sqrt{2x-1} - 1 = \sqrt{x-1}$. Now you can't isolate both radicals, and it doesn't help to put them together on one side of the equation. So you do the best you can, with one of the radicals isolated in the equation as it stands. Squaring, you get

$$2x - 1 - 2\sqrt{2x-1} + 1 = x - 1$$

Now we can isolate the remaining radical:

$$2\sqrt{2x-1} = x + 1$$

Squaring again, and rearranging, gives us a quadratic equation:

$$x^2 - 6x + 5 = 0$$

with roots $x = 1$ or $x = 5$.

Warning! Squaring may introduce so-called *extraneous* roots, because, for example, if we started with $x = 3$ and squared it, we would have $x^2 = 9$. Taking the square root of this would appear to allow $x = -3$ as well as $x = 3$. So always check your final results to see if they fit the equation in its original form.

Exercise 2.7.0:
Solve for x:

1. $\sqrt{2x-1} = x - 2$ (Be careful!)
2. $\sqrt{x-3} = \sqrt{2x-5} - 1$

2.7.2 Combining Fractions

1. Multiplying: $\frac{a}{b}\frac{c}{d} = \frac{ac}{bd}$
2. Dividing: $\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b}\frac{d}{c} = \frac{ad}{bc}$
3. Adding: $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$
4. Subtracting: $\frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}$

The quantities a, b, c, d may be ordinary numbers, but they may also be algrebraic expressions. In either case, look first to see if they can be factored. In (a) and (b) this may lead to simple cancellation; in (c) and (d) it may enable you to find a least common denominator that is simpler than the product bd. In the latter case, convert each fraction to one with this common denominator and then add and subtract the numerators as required. This may involve less complexity than if you mechanically apply the above expressions.

Exercise 2.7.1:
Evaluate the following combinations of fractions, bringing the result to a single denominator in each case and cancelling where possible:

1. $\left(\dfrac{x-y}{x^2+y^2}\right)\left(\dfrac{y}{y-x}\right)$
2. $\dfrac{\left(\frac{x^2y+y^3}{2x}\right)}{\left(\frac{2xy+y^2}{4x^3}\right)}$
3. $\dfrac{x}{4y^2x}+\dfrac{z^2}{8xy}$
4. $\dfrac{4z}{xy^2}-\dfrac{2x}{y^3x}+\dfrac{z}{x^2y}$

3 Answers to Exercises

Answers to exercise 2.1 not provided.

Exercise **??**: (a) $8.0516 \cdot 10^4$; (b) $7.51 \cdot 10^{-2}$; (c) $3.52 \cdot 10^6$; (d) $8.1 \cdot 10^{-8}$

Exercise 2.1: (a) 1.125; (b) $1.39 \cdot 10^4$; (c) $2 \cdot 10^6$

Exercise 2.1: (a) $1.051 \cdot 10^{10}$; (b) $4.65 \cdot 10^5$; (c) $-1.429 \cdot 10^{26}$; (d) $3.95 \cdot 10^{-19}$; (e) $6.72 \cdot 10^{-9}$

Exercise 2.3.1: (a) $x = \frac{-11}{24}$; (b) $x = \frac{30}{17}$; (c) $x = \frac{c-3b}{3a-5b}$

Exercise 2.3.2: (a) $x = \frac{7}{5}$, $y = -\frac{2}{5}$; (b) $a = -\frac{8}{9}$, $b = \frac{14}{9}$; (c) $x = 1$, $y = 2$

Exercise 2.4: $6x^3 + 5x^2 - 14x + 5$

Exercise 2.5.2: (a) $6, -2$; (b) $\frac{1\pm\sqrt{2}}{3}$; (c) $\frac{-1\pm\sqrt{6}}{4}$; (d) $\frac{-3\pm\sqrt{15}}{2}$

Exercise 2.5.2: (a) $\frac{9\pm\sqrt{17}}{4}$; (b) $\frac{u\pm\sqrt{u^2-2gy}}{g}$; (c) $\frac{-6\pm\sqrt{46}}{5}$; (d) $\frac{15\pm\sqrt{465}}{4}$; (e)

$s \pm \sqrt{1-s^2}$

Exercise 2.6: $5x+4$

Exercise 2.6: The degree of the remainder is zero. (The remainder is $-2 = -2x^0$.)

Exercise 2.6: Degree of quotient: 3; degree of remainder: 1 (at most).

Exercise 2.6: You can't do it by long division. We assumed that the divider had a lower degree than the dividend, in order for the long division to work. It's like trying to divide 2 by 7. So the answer is (again, without uusing long division!): quotient $= 0$; remainder = dividend.

Exercise 2.6: (a) $(4x-3)(2x+5)$; (b) No (complex roots); (c) $(3x-4a)^2$; (d) $2\left(x+\frac{5}{2}-\frac{\sqrt{137}}{2}\right)\left(x+\frac{5}{2}+\frac{\sqrt{137}}{2}\right)$; (e) $100\left(x^2+10^4\right)(x+100)(x-100)$

Exercise 2.6: (a) $(x-2)(x-3)=0 \to 2, 3$; (b) $(2x-3)(x+4)=0 \to \frac{3}{2}, -4$; (c) $x(3x-5)=0 \to \frac{5}{3}, 0$; (d) $(2x-5)(3x+4)=0 \to \frac{5}{2}, -\frac{4}{3}$

Exercise 2.7.1: (a) $x = 5$ ($x = 1$ doesn't work, since in the original equation the LHS is a radical, and thus greater than 0; and the RHS is $1-2=-1<0$); (b) $x = 3$ or 7

Exercise 2.7.2: (a) $\frac{-y}{x^2+y^2}$; (b) $\frac{2x^2\left(x^2+y^2\right)}{2x+y}$; (c) $\frac{2x^2+yz^3}{8xy^2z}$; (d) $\frac{4xyz^2-2x^3+y^2z^2}{x^2y^3z}$

This module is based in large part on an earlier module prepared by the Department of Mathematics.

4 Algebra Review Problems

??, ?? Calculations

Problem 0: Express as a single number ,in scientific notation $a \times 10^k$, $1 \le a < 10$:

1. $\frac{(.024)(3\times10^{-2})}{8\times10^{12}}$

2. $\frac{1600}{32\times10^{-4}}$

Problem 1: 6.25×10^{24} molecules of water fill a .2 liter glass. Approximately how much of this volume (in liters) does one molecule account for?

Give your answer in scientific notation with the correct number of significant figures.

Problem 2: Using $E = mc^2$, where $c = 3 \times 10^8$ meters/sec, in $km \cdot sec$ units how much energy E is mass-equivalent to 3×10^{-18} kg of water?

Problem 3: Looking at a representative .60 ml sample under her microscope, Michelle counts exactly 120 bacteria. If the sample is drawn from a flask containing 2000. ml of water, how many bacteria are expected to be in the flask?

?? Linear equations
Problem 4: Solve for x in terms of a and b: $\frac{x+a}{x-a} = b$.

Problem 5: Solve simultaneously for x and y:

1.

$$\begin{aligned} 2x - 3y &= -1 \\ 3x - 2y &= 6 \end{aligned}$$

2.

$$\begin{aligned} 4x - 3y &= 9 \\ 5y - 3x - 7 &= 0 \end{aligned}$$

3.

$$\begin{aligned} x + 2y &= a \\ x - y &= b \end{aligned}$$

Problem 6: Brown rice comes in 5 lb bags costing \$2 a bag; wild rice comes in 2 lb. bags costing \$10 a bag. Egbert has just spent \$32 buying 34 lbs. of rice for his commune. How many bags of each did he buy?

Problem 7: How many liters of liquid endersol should be added to 20 liters of water to get a solution that is 40% endersol?

?? Polynomials, binomial theorem

The general *binomial theorem* is something you should know. It states that

$$(a+b)^n = a^n + na^{n-1}b + \cdots + \binom{n}{k} a^{n-k}b^k + \cdots + nab^{n-1} + b^n$$

where the coefficient of the general term is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$$

The cases $n = 2$ and $n = 3$ are given in the review modules.

Problem 8: Write $(x+1)^3$ as a polynomial in x using the binomial theorem.

Problem 9: Simplify $\frac{(x+h)^3 - x^3}{h}$, writing it as a polynomial in x and h.

Problem 10: What are the coefficients a, b, c in $(x+1)^1 2 = ax^1 2 + bx^1 1 + cx^1 0 + \cdots$?

Problem 11: Write $(x+y)^4$ as a polynomial in x and y.

?? Quadratic equations

Try a selection of these problems; similar problems are grouped together. If the coefficient of x^2 is not 1, be especially careful in using the quadratic formula.

Problem 12: For each of the following, first try to find the roots by factoring the left hand side. If successful, then use the quadratic formula to check yoru answer; if not, find the roots by the quadratic formula. Try a, c, and d, b for more practice.

1. $x^2 - 7x + 12 = 0$
2. $x^2 + 2x - 35 = 0$
3. $x^2 - 6x + 5 = 0$
4. $2x^2 - 5x + 3 = 0$

Problem 13:

1. Solve for t in terms of the other variables: $y = vt - \frac{1}{2}gt^2$.
2. Express x in terms of a, b, and h, if $h = 2ax - bx^2$ and a, b, h, $x > 0$.

Problem 14: Solve $x^2 + 3x - 1 = 0$, and tell whether both roots are positive, negative, or of opposite signs. How could you have answered this last question without first solving the equation?

Problem 15: Let a, b be the distinct roots of $x^2 - 4x + 2$, with $a < b$. Which correctly compares a and b to 1: $a < b < 1$, $a < 1 < b$, or $1 < a < b$?

Problem 16: For which values of the constant b will the roots be real:

1. $x^2 + 4x - b = 0$

2. $x^2 + bx + 1 = 0$

Problem 17: A rectangle has area 7, and the width is 2 less than the height. What are its dimensions?

Problem 18: The sum of three times a certain number and twice its reciprocal is 5. Find all such numbers.

Problem 19: A rock is thrown off the ledge of a 100 foot cliff at time $t = 0$. The height of the rock above the ground is then given in terms of t (in seconds) by the formula $40t - 16t^2 + 100$. After how many seconds does the rock hit the ground?

?? Factoring

Problem 20: Factor each of the following, using all information given:

1. $f(x) = x^3 - 2x^2 - x + 2$, $f(-1) = 0$
2. $f(x) = x^3 - 2x - 4$, $f(2) = 0$
3. $x^4 - 16$
4. $x^3 - x^2 - 9x + 9$

Problem 21: Does $x - 1$ divide evenly the polynomial $5x^8 - 3x^5 + x^4 - 2x - 1$? Hint: do not actually perform any division!

Problem 22: For what value(s) of the constant c will:

1. $x - 1$ be a factor of $x^3 - 3x^2 + cx - 2$?

2. $x+2$ be a factor of x^4+4x^3+c?

?? Solving other types of equations: algebraic manipulations Problem 23: Solve for x:

1. $\sqrt{3x+10} = x+2$
2. $\sqrt{1-x^2} = x/\sqrt{3}$

Problem 24: Express x in terms of y, if $\frac{1+x^2}{1-x^2} = y$.

Problem 25: Express m in terms of c and a: $(m^2-c^2)^{-1/2} = a$, given that $m > 0$.

Problem 26: Combine the two terms:

1. $\frac{a}{a+h} + \frac{h}{a-h}$
2. $\frac{x}{x+1} + \frac{2}{x-2}$

Problem 27: Solve: $3(x+1)^{-1} + (x-3)^{-1} = 1$.

Problem 28: For each postiive integer n, a_n is given: evaluate and simplify a_{n+1}/a_n:

1. $a_n = \frac{x^n}{n!}$
2. $a_n = \frac{x^{2n}}{n(n+1)}$

?? Geometric series and geometric progressions
The formula for the sum of the *infinite geometric series* is:

$$1 + r + r^2 + r^3 + \cdots + r^n + \cdots = \frac{1}{1-r}$$

, if $-1 < r < 1$.

If the series is stopped after n terms, we get what is called a *geometric sum* or *geometric progression*; its sum is given by

$$1 + r + r^2 + r^3 + \cdots + r^n = \frac{1-r^{n+1}}{1-r}$$

In the above formulas, both sides may be multiplied by a constant factor a, givuing a formula for the sum of $a + ar + ar^2 + \cdots$. It's easiest to remember the formulas in the above form, however, and if you remember the trick in problem 4 below, you can even get away with just remembering the first formula, for the sum of the series.

Problem 29: Find the sum of:

1. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$
2. $2 + \frac{2}{10} + \frac{2}{100} + \cdots$

Problem 30: Express each of teh following repeating decimals as a common fraction, by interpreting it as an infinite geometric series:

1. $.4444444444\ldots$
2. $.12121212\ldots$

Problem 31: A turtle travels along a road. The first day it covers 100 feet, and on each succeeding day it covers 2/3 the distance it traveled the day before. How far can it get in this way?

Problem 32: Prove the formula for the sum of a geometric progression by multiplying the equation by a constant.

Problem 33: Derive the second formula from the first, with the same bounds on r, by writing

$$1 + r + \cdots + r^n = (1 + r + \cdots + r^n + \cdots) - r^{n+1}(1 + r + \cdots + r_n + \cdots)$$

5 Solutions to Algebra Review Problems

Solution to problem 0:

1.
$$\frac{(.024)(3 \times 10^{-2})}{8 \times 10^{12}} = \frac{24 \times 10^{-3} \times 3 \times 10^{-2}}{8 \times 10^{12}} = \left(\frac{24 \cdot 3}{8}\right) 10^{-5} \cdot 10^{-12} = \boxed{9 \times 10^{-17}}$$

2.
$$\frac{1600}{32 \times 10^{-4}} = \frac{16 \times 10^2}{32 \times 10^{-4}} = .5 \times 10^6 = \boxed{5 \times 10^5}$$

Solution to problem 1: Let x be the volume accounted for by one molecule. Then

$$\frac{6.25 \times 10^{24}}{.2} = \frac{1}{x} \longrightarrow x = \frac{.2}{6.25 \times 10^{24}} = \frac{2}{6.25} \times \frac{10^{-1}}{10^{24}} = .3 \times 10^{-25} = \boxed{3 \times 10^{-26}}$$

Solution to problem 2: $E = (3 \times 10^{-18})(3 \times 10^8)^2 = 27 \times 10^{-18} \times 10^{16} = \boxed{2.7 \times 10^{-1}}$.

Solution to problem 3: $\frac{.60}{120} = \frac{2000}{x} \longrightarrow x = \frac{2000}{.60} \times 120 = \frac{2 \times 12 \times 10^3 \times 10}{6 \times 10^{-1}} = \boxed{4 \times 10^5}$

Solution to problem 4: Cross-multiply $\frac{x+a}{x-a} = b$ to get

$$
\begin{aligned}
x + a &= bx - ba \\
x - bx &= -a - ba \\
x &= -\frac{a(1+b)}{1-b} \\
x &= \boxed{a(b+1)}b - 1
\end{aligned}
$$

Solution to problem 5:

1. We have

$$
\begin{aligned}
2x - 3y &= -1 \\
3x - 2y &= 6
\end{aligned}
$$

To eliminate y, we multiply the top by 2 and bottom by 3 and subtract:

$$
\begin{aligned}
4x - 6y &= -2 \\
9x - 6y &= 18 \\
-5x &= -20
\end{aligned}
$$

$\boxed{x = 4, y = 3}$.

2.

$$
\begin{aligned}
4x - 3y &= 9 \\
-3x + 5y &= 7
\end{aligned}
$$

Multiply top by 5, bottom by 3, and add:

$$
\begin{aligned}
20x - 15y &= 45 \\
-9x + 15y &= 21 \\
11x &= 66
\end{aligned}
$$

$\boxed{x = 6, y = 5}$.

3.

$$\begin{aligned} x + 2y &= a \\ x - y &= b \\ 3y &= a - b \\ y &= \frac{a-b}{3} \end{aligned}$$

$x = \frac{a+2b}{3}, y = \frac{a-b}{3}$.

Solution to problem 6: Let x be the number of bags of brown rice, y be the number of bags of wild rice; then we have the system of equations:

$$\begin{aligned} 2x + 10y &= 32 \\ 5x + 2y &= 34 \end{aligned}$$

Eliminate y be multiplying bottom equation by 5 and subtracting: $-23x = -138 \rightarrow \boxed{x = 6, y = 2}$.

Solution to problem 7: x is the number of liters of endersol.

$$\frac{x}{x+20} = \frac{40}{100} = \frac{2}{5} \rightarrow 6x = 2x + 40 \rightarrow \boxed{x = 40/3}$$

Solution to problem 8:

$$\begin{aligned} (x+1)^3 &= x^3 + \binom{3}{1} x^2 + \binom{3}{2} x + \binom{3}{3} \\ &= x^3 + 3x^2 + 3x + 1 \end{aligned}$$

Solution to problem 9:

$$\begin{aligned} \frac{(x+h)^3 - x^3}{h} &= \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\ &= 3x^2 + 3xh + h^2 \end{aligned}$$

Solution to problem 10: Note that $\binom{12}{2} = \frac{12 \cdot 11}{2} = 6 \cdot 11 = 66$.

$$\begin{aligned} (x+1)^{12} &= x^12 + \binom{12}{1} x^11 + \binom{12}{2} x^{10} + \cdots \\ &= x^12 + 12x^11 + 66x^10 + \cdots \end{aligned}$$

Solution to problem 11: $(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$ (Note: $\binom{4}{2} = \frac{4\cdot 3}{2} = 6$.)

Solution to problem 12:

1. $x^2 - 7x + 12) = (x-3)(x-4) = 0 \to \boxed{x = 3, 4}$
2. $x^2 + 2x - 35 = (x+7)(x-5) = 0 \to \boxed{x = -7, 5}$
3. $x^2 - 5x + 5$ doesn't factor. By formula: $x = \frac{5 \pm \sqrt{4+4\cdot 35}}{2} = \frac{5\pm\sqrt{5}}{2}$.
4. $2x^2 - 5x + 3 = (2x-3)(x-1) = 0 \to \boxed{x = 1, \dfrac{3}{2}}$.

Solution to problem 13:

1. $y = vt - \frac{1}{2}gt^2$ is a quadratic in t.

   Rewrite as $\frac{1}{2}gt^2 - vt + y = 0$; by formula, $t = \frac{v \pm \sqrt{v^2 - 4\cdot\frac{g}{2}y}}{2\cdot g/2} = \frac{v\pm\sqrt{v^2-2gy}}{g}$.
2. $h = 2ax - bx^2$ is a quadratic in x; rewrite as $bx^2 - 2ax + h = 0$, so by formula, $x = \frac{2a\pm\sqrt{4a^2-4bh}}{2b} = \frac{a\pm\sqrt{a^2-bh}}{b}$.

Solution to problem 14: $x^2 + 3x - 1 = 0$, so by the quadratic formula, $x = \frac{-3\pm\sqrt{9-4(-1)}}{2} = \boxed{-3 \pm \sqrt{13}}2$.

Since $3 < \sqrt{13}$, we see that $-3 + \sqrt{13} > 0$, while $-3 - \sqrt{13} < 0$. Therefore, the roots are of opposite signs.

Alternately, since $x^2 + 3x - 1 = (x - r_1)(x - r_2)$, we have $r_1 r_2 = -1$, which shows that roots have opposite signs without actually calculating them.

Solution to problem 15: $x^2 - 4x + 2 = 0$, by formula: $x = \frac{-4\pm\sqrt{16-4\cdot 2}}{2} = 2 \pm \sqrt{2}$. Since $\sqrt{2} \approx 1.4$, $2 - \sqrt{2} < 1 < 2 + \sqrt{2}$.

Solution to problem 16:

1. $x^2 + 4x - b = 0$ by formula gives $x = \frac{-4\pm\sqrt{16+4b}}{2}$, so roots are real iff (if and only if) $16 + 4b \geq 0$ or $\boxed{b \geq -4}$.
2. $x^2 + bx + 1 = 0$ by formula gives $x = \frac{-b\pm\sqrt{b^2-4}}{2}$, so roots are real iff $b^2 - 4 \geq 0 \to b^2 \geq 4$, so iff $b \geq 2$ or $b \leq -2$.

Solution to problem 17: Let x be the width, y be the height. We have

$$\begin{aligned} y - 2 &= x \\ xy &= 7 \end{aligned}$$

Substitution into the second gives

$$\begin{aligned} (y-2)y &= 7 \\ y^2 - 2y - 7 &= 0 \\ y &= \frac{2 \pm \sqrt{4+28}}{2} = 2\sqrt{2}+1 \\ x &= 2\sqrt{2}-1 \end{aligned}$$

Solution to problem 18: Let x be the number. Then $3x + \frac{2}{x} = 5 \rightarrow 3x^2 - 5x + 2 = 0$. By formula,

$$x = \frac{5 \pm \sqrt{26 - 4 \cdot 2 \cdot 3}}{6} = \frac{5 \pm 1}{6} = \boxed{1, \frac{2}{3}}$$

Solution to problem 19: Solve $16t^2 - 40t - 100 = 0$, or $4t^2 - 10t - 25 = 0$: this gives $t = \frac{10 \pm \sqrt{100+4 \cdot 25 \cdot 4}}{2 \cdot 4} = \frac{10 \pm 10\sqrt{5}}{8} = \boxed{\frac{5}{4}\left(1+\sqrt{5}\right)}$. We reject the other solution because it makes no physical sense.

Solution to problem 20:

1. $f(-1) = 0$ means that $x+1$ is a factor. Thus, $x^3 - 2x^2 - x + 2 = (x+1)(x^2 - 3x + 2) = (x+1)(x-2)(x-1)$. (We get the factor $x^2 - 3x + 2$ by long division.)

2. $f(2) = 0$ means that $x - 2$ is a factor; $x^3 - 2x - 4 = (x-2)(x^2 + 2x + 2)$.

3. $x^4 - 16 = (x^2 - 4)(x^2 + 4) = (x-2)(x+2)(x^2+4)$.

4. $x^3 - x^2 - 9x + 9$ — We try as roots the factors of 9, and see that 1 is a root, so $x - 1$ is a factor. $x^3 - x^2 - 9x + 9 = (x-1)(x^2 - 9) = (x-1)(x-3)(x+3)$.

Solution to problem 21: $x-1$ divides $f(x)$ means that $f(1)=0$. We calculate therefore $f(1): 5\cdot 1^8-3\cdot 1^5+1^4-2\cdot 1-1=0$, so $x-1$ is a factor.

Solution to problem 22:

1. $x-1$ is a factor implies $f(1)=0$; $1^3-3\cdot 1^2+c\cdot 1-2=0$ so $c-4=0\to \boxed{c=4}$.

2. $x+2$ is a factor implies $f(-2)=0$; $(-2)^4+4(-2)^3+c=0\to 16-32+c=0\to \boxed{c=16}$.

Solution to problem 23:

1.

$$\begin{aligned}\sqrt{3x+10}&=x+2\\ 3x+10&=x^2+4x+4\\ x^2+x-6&=0\\ (x+3)(x-2)&=0\end{aligned}$$

We reject $x=-3$, since it doesn't satisfy the equation we started with; thus the only solution is $\boxed{x=2}$.

2.

$$\begin{aligned}\sqrt{1-x^2}&=\frac{x}{\sqrt{3}}\\ 1-x^2&=\frac{x^2}{3}\\ 1&=\frac{4}{3}x^2\\ x&=\sqrt{\frac{3}{4}}=\boxed{\frac{\sqrt{3}}{2}}\end{aligned}$$

Again, we reject $-\sqrt{3}/2$ since it doesn't work in the original equation.

Solution to problem 24: $\frac{1+x^2}{1-x^2} = y$ – solve for x^2: $1 + x^2 = y - x^2 y \rightarrow x^2(1+y) = y - 1 \rightarrow x^2 = \frac{y-1}{y+1}$, so

$$\boxed{x = \pm\sqrt{\frac{y-1}{y+1}}}$$

Solution to problem 25:

$$\begin{aligned}\frac{1}{\sqrt{m^2 - c^2}} &= a \\ \frac{1}{m^2 - c^2} &= a^2 \\ \frac{1}{a^2} &= m^2 - c^2 \\ m^2 &= c^2 + \frac{1}{a^2} \\ m &= \sqrt{c^2 + \frac{1}{a^2}}\end{aligned}$$

Solution to problem 26:

1. $$\frac{a}{a+h} + \frac{h}{a-h} = \frac{a(a-h) + h(a+h)}{(a+h)(a-h)} = \frac{a^2 + h^2}{a^2 - h^2}$$

2. $$\frac{x}{x+1} + \frac{2}{x-2} = \frac{x^2 + 2}{(x+1)(x-2)}$$

Solution to problem 27:

$$\frac{3}{x+1} + \frac{1}{x-3} = 1 \rightarrow \frac{4x-8}{x^2 - 2x - 3} = 1 \rightarrow 4x - 8 = x^2 - 2x - 3 \rightarrow x^2 - 6x + 5 = 0$$

This last factors to $(x-5)(x-1) = 0$, so $\boxed{x = 1, 5}$ are the solutions.

Solution to problem 28:

1. $\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} = \boxed{\dfrac{x}{n+1}}$

2. $\frac{a_{n+1}}{a_n} = \frac{x^{2(n+1)}}{(n+1)(n+2)} \cdot \frac{n(n+1)}{x^{2n}} = \boxed{x^2 \cdot \dfrac{n}{n+2}}$

Solution to problem 29:

1. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} = \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{1}{3/2} = \boxed{\dfrac{2}{3}}\ (r = -\frac{1}{2})$

2. $2 + \frac{2}{10} + \frac{2}{100} + \cdots = 2 \cdot \frac{1}{1-\frac{1}{10}} = \boxed{\dfrac{20}{9}}\ (r = \frac{1}{10})$

Solution to problem 30:

1. $.444\cdots = \frac{4}{10} + \frac{4}{100} + \frac{4}{1000} + \cdots = \frac{4}{10}\left(1 + \frac{1}{10} + \frac{1}{100} + \cdots\right) = \frac{4}{10}\left(\frac{1}{1-\frac{1}{10}}\right) =$ $\boxed{\dfrac{4}{9}}$

2. $.121212\cdots = \frac{12}{100} + \frac{12}{10^4} + \frac{12}{10^6} + \cdots = \frac{12}{100} \cdot \frac{1}{1-1/100} = \boxed{\dfrac{12}{99}}$

Solution to problem 31:

$$100 + \frac{2}{3}100 + \frac{2}{3} \cdot \frac{2}{3} \cdot 100 + \cdots = 100\left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots\right) = 100 \cdot \frac{1}{1-2/3} = \boxed{300}$$

Solution to problem 32: $(1 + r + \cdots + r^n)(1-r) = (1 + r + \cdots + r^n) - (r + r^2 + \cdots + r^{n+1}) = 1 - r^{n+1}$.

Solution to problem 33:

$$\begin{aligned} 1 + r + \cdots + r^n &= (1 + r + r^2 + \cdots) - r^{n+1}(1 + r + \cdots) \\ &= \frac{1}{1-r} - \frac{r^{n+1}}{1-r} \\ &= \frac{1-r^{n+1}}{1-r} \end{aligned}$$

6 Algebra Diagnostic Test #1

Problem 34: Evaluate and express in scientific notation with the corret number of significant figures $\frac{1}{8 \cdot 10^{12}}$ and $\frac{160}{32 \cdot 10^{-4}}$.

Problem 35: Jack and Janet are buying an assortment of fruit. Jack wants the number of apples to be three lessw than twice the number of oranges. Janet wants to buy two more apples than oranges. How many oranges and apples should they get?

Problem 36: Expand the following polynomial and determine its degree:

$$x^4 - 3x^3 + 3x^2 + (2 - x^2)(x^2 + 2)$$

Problem 37: A man throws a rock off the ledge of a 100 foot high cliff at time $t = 0$. The height of the rock above ground level is then given in terms of the time t in seconds by $40t - 16t^2 + 100$. After how many seconds does the rock hit the ground?

Problem 38: Factor $x^3 - x^2 - 16x - 16$.

Problem 39: Simplify

$$\frac{(z + h)^3 - z^3}{h}$$

7 Algebra Diagnostic Test Solutions

Problem 40: Evaluate and express in scientific notation with the correct number of significant figures $\frac{1}{8\cdot 10[12}$ and $\frac{160}{.32\cdot 10^{-4}}$.

Solution to problem 34:

$$\frac{1}{8\cdot 10^{12}} = \frac{1}{8}\times 10^{-12} = .125\times 10^{-12} = \boxed{1.3\times 10^{-13}}$$

$$\frac{160}{.32\times 10^{-4}} = \frac{16\times 10^{1}}{32\times 10^{-6}} = .5\times 10^{7} = \boxed{5\times 10^{6}}$$

Problem 41: Jack and Janet are buying an assortment of fruit. Jack wants the number of apples to be three lessw than twice the number of oranges. Janet wants to buy two more apples than oranges. How many oranges and apples should they get?

Solution to problem 35: Let x be the number of apples, y be the number of oranges. Then we have the system of equations

$$\begin{aligned} x &= 2y - 3 \\ x &= y + 2 \end{aligned}$$

Solve simultaneously. Subtracting, $0 = y - 5$ so $y = 5$, $x = 7$. Answer: $\boxed{5 oranges, 7 apples}$.

Problem 42: Expand the following polynomial and determine its degree:

$$x^4 - 3x^3 + 3x^2 + (2 - x^2)(x^2 + 2)$$

Solution to problem 36:

$$\begin{aligned} &= x^4 - 3x^3 + 3x^2 + (2x^2 + 4 - x^4 - 2x^2) \\ &= -3x^3 + 3x^2 + 4 \end{aligned}$$

$\boxed{degree3}$

Problem 43: A man throws a rock of the ledge of a 100 foot high cliff at time $t = 0$. The height of the rock above ground level is then given in terms of the time t (seconds) by $40t - 16t^2 + 100$. After how many seconds does the rock hit the ground?

Solution to problem 37: The problem states the height at time t is $40t - 16t^2 + 100$, and asks what is t when the height is 0?

Thus, we solve $-16t^2 + 40t + 100 = 0 \to 4t^2 - 10t - 25 = 0$. This gives

$$t = \frac{10 \pm \sqrt{100 - 4 \cdot 4 \cdot (-25)}}{2 \cdot 4} = \frac{10 \pm \sqrt{500}}{8} = \frac{10}{8}\left(1 + \sqrt{5}\right)$$

as we reject the negative root. Thus, $\boxed{t = \frac{5}{4}\left(1 + \sqrt{5}\right)}$.

Problem 44: Factor $x^3 - x^2 - 16x + 16$.

Solution to problem 38: Let $f(x) = x^3 - x^2 - 16x + 16$. By inspection, $f(1) = 0$, so $x - 1$ is a factor.

Now $f(x) = (x-1)(x^2 - 16)$ (determine $x^2 - 16$ by division or inspection); this factors further to $\boxed{(x-1)(x-4)(x+4)}$.

Problem 45: Simplify $\frac{(x+h)^3 - x^3}{h}$.

Solution to problem 39:

$$\begin{aligned}\frac{(x+h)^3 - x^3}{h} &= \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\ &= \boxed{3x^2 + 3xh + h^2}\end{aligned}$$

8 Algebra Diagnostic Test #2

Problem 46: 6.25×10^{24} molecules of water fill a .2 liter glass. Approximately how much of this volume in liters does one molecule account for?

Problem 47: Solve for x and y in terms of a and b, simplifying your answer as much as possible:

$$\begin{aligned} x + 2y &= a \\ x - y &= b \end{aligned}$$

Problem 48: Write $(x+1)^3$ as a polynomial in x.

Problem 49: SUppose a and b are the distinct roots of $x^2 - 4x - 2$, with $a < b$. Which of the following correctly compares a and b to 1?

1. $a < b < 1$
2. $a < 1 < b$
3. $a < b < 1$

Problem 50: Does $x - 1$ divide evenly into $5x^8 - 3x^5 + x^4 - 2x - 1$? Hint: do not attempt to divide!

Problem 51: First solve for x^2, then for x:

$$\frac{1+x^2}{1-x^2} = y$$

9 Algebra Diagnostic Test #2 Solutions

Problem 52: 6.25×10^{24} molecules of water fill a .2 liter glass. Approximately how much of this volume in liters does one molecule account for?

Solution to problem 40:

$$\frac{.2}{6.25 \times 10^{24}} = \frac{20 \times 10^{-2}}{6.25 \times 10^{24}} = 3 \times 10^{-26}$$

(with one significant figure)

Problem 53: Solve for x and y in terms of a and b, simplifying your answer as much as possible:

$$\begin{aligned} x + 2y &= a \\ x - y &= b \end{aligned}$$

Solution to problem 41: Subtracting: $3y = a - b$, so $\boxed{y = \frac{a-b}{3}}$.

Substituting: $x = y + b$, so $\boxed{x = \frac{a+2b}{3}}$.

Problem 54: Write $(x+1)^3$ as a polynomial in x.

Solution to problem 42: $(x+1)^3 = x^3 + 3x^2 + 3x + 1$ by the binomial theorem.

Problem 55: SUppose a and b are the distinct roots of $x^2 - 4x - 2$, with $a < b$. Which of the following correctly compares a and b to 1?

1. $a < b < 1$
2. $a < 1 < b$
3. $a < b < 1$

Solution to problem 43: The solutions to $x^2 - 4x + 2 = 0$ are $x = \frac{4 \pm \sqrt{16-4\cdot 2}}{2} = \frac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2}$, the roots. Thus $a = 2 - \sqrt{2} \approx .6$, and $b = 2 + \sqrt{2} \approx 3.4$, so the correct comparison is $\boxed{a < 1 < b}$.

Problem 56: Does $x - 1$ divide evenly into $5x^8 - 3x^5 + x^4 - 2x - 1$? Hint: do not attempt to divide!

Solution to problem 44: $x - 1$ is a factor of $f(x)$ is equivalent to $f(1) = 0$. But plugging in $x = 1$, $5 - 3 + 1 - 2 - 1 = 0$, so $x - 1$ is a factor.

Problem 57: First solve for x^2, then for x:

$$\frac{1 + x^2}{1 - x^2} = y$$

Solution to problem 45: Solving:

$$\begin{aligned}
1 + x^2 &= y - x^2 y \\
x^2(y + 1) &= y - 1 \\
x^2 &= \frac{y - 1}{y + 1} \\
x &= \pm\sqrt{\frac{y - 1}{y + 1}}
\end{aligned}$$

10 Self-Evaluation Summary

You may want to informally evaluate your understanding of the various topic areas you have worked through in the *Self-Paced Review*. If you meet with tutors, you can show this evaluation to them and discuss whether you were accurate in your self-assessment.

For each topic which you have covered, grade yourself on a one to ten scale. One means you completely understand the topic and are able to solve all the problems without any hesitation. Ten means you could not solve any problems easily without review.

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