MA 100 SAMPLE FINAL

Please write your name, your ID number and the color of your exam paper on your blue book!

Please try to show your work and give justifications for your answers. It is permitted to use calculators, graph paper, rulers, both sides of an 8 1/2 x 11 sheet of notes and a magnifying glass on the final. Try not to spend too much time on any single problem; if you get stuck on a problem leave a partial answer and move on to the next.

(1) (5 pts) Simplify: $\frac{x}{x-3} - \frac{1}{2x+1}$

$\frac{x}{x-3} - \frac{1}{2x+1} = \frac{x}{x-3} \cdot \frac{2x+1}{2x+1} - \frac{1}{2x+1} \cdot \frac{x-3}{x-3} = \frac{x(2x+1)}{(x-3)(2x+1)} - \frac{x-3}{(x-3)(2x+1)} = \frac{(2x^2+x)-(x-3)}{2x^2+x-6x-3} = \frac{2x^2+3}{2x^2-5x-3}$

(2) (5 pts) Compute: $(3-5i)(4+2i)$

$(3-5i)(4+2i) = 12 + 6i - 20i - 10i^2 = 12 - 14i - 10(-1) = 22 - 14i$

(3) (5 pts) For what values of x is $|x-5| > 2$?

i) x is more than two units from 5, so $x < 3$ or $x > 7$.

ii) $|x-5| > 2$, so either $-(x-5) > 2$ or $x - 5 > 2$. The first inequality simplifies as follows: $-x + 5 > 2, -x > -3, x < 3$. The second inequality simplifies to $x > 7$.

(4) (5 pts) Sketch the graph of $f(x) = |x+2|$. Label your sketch.

The graph is v-shaped with the vertex of the v at the point $(0,2)$. Be sure to label the x- and y-axes and the units on each axis.

(5) (5 pts) Give the equation of the line that passes through the points $(2,1)$ and $(8,3)$.

Slope formula: $m = \frac{\text{rise}}{\text{run}} = \frac{y_2-y_1}{x_2-x_1}$
$m = \frac{6}{2} = 3$
Point slope formula: $y - y_1 = m(x - x_1)$
$y - 1 = 3(x-2)$, so $y - 1 = 3x - 6$ or $y = 3x - 5$.

(6) (10 pts) Find all zeros (real and complex) of the polynomial $2x^4 + 2x^2 = 0$.

$2x^4 + 2x^2 = 2x^2(x^2+1)$ so $x = 0$ (with multiplicity 2) or $x^2 + 1 = 0$. If $x^2 + 1 = 0$ then $x^2 = -1$ and $x = \pm\sqrt{-1} = \pm i$ (alternately, use the quadratic formula on $x^2 - 1$.)

Answer: $x = 0$, $x = i$ or $x = -i$.

(7) (5 pts) Check your solutions to the problem above in the original equation.

i) $2(0)^4 + 2(0)^2 = 2 \cdot 0 + 2 \cdot 0 = 0 + 0 = 0$ ✓
ii) $2(i)^4 + 2(i)^2 = 2 \cdot 1 + 2(-1) = 2 - 2 = 0$ ✓
ii) $2(-i)^4 + 2(-i)^2 = 2 \cdot 1 + 2(-1) = 2 - 2 = 0$ ✓

(8) (10 pts) Describe the graph of the equation $(x-1)^2 + y^2 = 1$. (A labeled sketch of the graph counts as a description.)

The equation for a circle with center (h, k) and radius r is:

$$(x-h)^2 + (y-k)^2 = r^2$$

This graph is shaped like a circle of radius 1 whose center is at $(1, 0)$.

(9) (10 pts) Let $f(x) = e^{2x}$. What is $f^{-1}(x)$?

i) Replace $f(x)$ with y: $y = e^{2x}$.
ii) Swap x and y (to reverse the roles of input and output): $x = e^{2y}$. We want to solve for y in terms of x. Since y is "trapped" in an exponent, we need to apply the inverse of the exponential function to both sides of the equation. The inverse of the function e^u is $ln(u)$.
iii) $ln(x) = ln(e^{2y})$, which simplifies to $ln(x) = 2y$.
iv) Solve for y: $y = \frac{1}{2}ln(x)$.
v) Replace y with $f^{-1}(x)$: $f^{-1}(x) = \frac{1}{2}ln(x)$.

(10) (10 pts) If $f(x) = x^2 - 1$ and $g(x) = x + 1$, what is $f(g(x))$?

$$f(g(x)) = (g(x))^2 - 1 = (x+1)^2 - 1 = (x+1)(x+1) - 1 = x^2 + 2x + 1 - 1 = x^2 + 2x.$$

(11) (10 pts) Compare the distance between the points (x_1, y_1) and (x_2, y_2) to the sum $|x_2 - x_1| + |y_2 - y_1|$. When is the distance greater than the sum? Equal to the sum? Less than the sum?

The easiest way to solve this is to realize that you are comparing the sum of the lengths of the legs of a right triangle (segments from (x_1, y_1) to (x_2, y_1) and from (x_2, y_1) to (x_2, y_2)) to the length of its hypotenuse (the segment from (x_1, y_1) to (x_2, y_2)). The distance between the points will be strictly less than $|x_2 - x_1| + |y_2 - y_1|$ unless either $|x_2 - x_1|$ or $|y_2 - y_1|$ is zero; i.e. when $x_1 = x_2$ or when $y_1 = y_2$.

The hardest way to solve this is to use the distance formula:

$$\text{distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

If you square the two quantities being compared you get

$$(x_2 - x_1)^2 + (y_2 - y_1)^2$$

from the distance formula and, from the other quantity,

$$|x_2 - x_1|^2 + 2|x_2 - x_1||y_2 - y_1| + |y_2 - y_1|^2 = (x_2 - x_1)^2 + 2|x_2 - x_1||y_2 - y_1| + (y_2 - y_1)^2.$$

Since the square of the sum of absolute values is greater than the square of the distance (by $2|x_2 - x_1||y_2 - y_1|$) you can conclude that the sum of the absolute values is greater than the distance except when $2|x_2 - x_1||y_2 - y_1| = 0$; i.e. when $x_1 = x_2$ or when $y_1 = y_2$.

The slowest way to solve this is to make a (large) table of possible values for x_1, x_2, y_1 and y_2 and compute the values being compared.

(12) (10 pts) Sketch the graph of $f(x) = \frac{x^2+x-7}{x^2-1}$. Label your sketch.

i) $f(0) = \frac{-7}{-1} = 7$

ii) $f(x) = 0$ when $x^2 + x - 7$. This is not simple to factor (because the "7" was meant to be a "6"), so we use the quadratic formula to find roots of $x^2 + x - 7 = 0$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1 - (-28)}}{2}$$

So $f(x) = 0$ when $x \cong -2.19$ or $x \cong 3.19$.

iii) $f(x)$ has vertical asymptotes where its denominator is zero. We know that $x^2 - 1 = 0$ when $x = -1$ and when $x = 1$.

iv) To get the "big picture", we divide the highest degree term in the numerator by the highest degree term in the denominator. $\frac{x^2}{x^2} = 1$, so there is a horizontal asymptote at $y = 1$.

v) We know how the function behaves when x is near $-2.19, -1, 0, 1, 3.19$, very large, or very large and negative. We compute the values of $f(x)$ at $-3, -2, -.5, .5, 2$ and 4 to help us graph the function.

x	$-\infty$	-3	-2.19	-2	-.5	0	.5	2	3.19	4	∞
$f(x)$	1	-0.125	0	-1.67	9	7	-8.33	-0.333	0	0.867	1

vi) Draw and label a coordinate system, sketch in the horizontal and vertical asymptotes, and plot the points. Then sketch the graph.

(13) (10 pts) Given the graph of a function but not its formula (e.g. one of the eight graphs shown on page 130), estimate the solutions of the equation $f(x) = 2$.

To find the values of x for which $f(x) = 2$ from the graph, find the point $(0, 2)$ on the y-axis (start at the origin and go up two units). Then look to the right and left for places where the y coordinate of a point on the graph of the function is 2; places where the graph of the function crosses the line $y = 2$.

For each of these points, find the x coordinate by drawing a vertical line down to the x-axis and using the scale marked on the x-axis. The values of x that you find in this way are the values of x for which $f(x) = 2$.

Approximate answers for the graphs shown on page 130:

a) $x = 3$, $x = -1$, $x = 1$ or $x = 3$
b) $x = -3$ or $x = 1$
c) $x = 1$
d) $x = -2.5$ or $x = 1$
e) $x = -2$
f) $x = -1.5$, $x = -0.5$ or $x = 1.5$
g) $x = -1$ or $x = 5$
h) $x = -2$ or $x = -.5$