18.100B Lecture Notes

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Thm

a) $\forall x \in \mathbb{R} \exists n \in \mathbb{Z}$ s.t. $n > x$

b) $\forall x, y \in R$ s.t. $x < y, \exists p \in \mathbb{Q}$ s.t. $x < p < y$

Pf We proved (a) proved last class
Choose a $n \in \mathbb{Z} > (y-x)^{-1} \Rightarrow \frac{1}{n} < y - x$
Now choose $m \in Z$ s.t. $m - 1 \leq nx < m$

$$\frac{m-1}{n} \leq x < \frac{m}{n}$$

$$\frac{m-1}{n} + \frac{1}{n} < x + y - x \Rightarrow \frac{m}{n} < y$$

Defn $f : A \to B$ is

injective (1:1) if $f(a_1) = f(a_2) \to a_1 = a_2$

surjective (onto) if $f^{-1}(b) \neq \emptyset \forall b \in B$

bijective if it is injective and surjective

Defn A and B can be put in 1-1 correspondence iff $\exists f : A \to B$ s.t. f is bijective.
We write $A \sim B$ to indicate this.
$\sim$ is reflexive, transitive, and symmetric. Thus, $\sim$ is an equivalence relation

Defn Let $J_n = \{1, 2, 3, ...n\}$. $J = \mathbb{N}$

a) A is finite iff $A \sim J_n$ or $A = \emptyset$

b) A is countable iff $A \sim J$

c) A is uncountable iff A is infinite and not countable

Ex $\mathbb{Z}$ is countable

Thm If A is countable, any infinite $E \subset A$ is countable

Pf We have a bijection $f : J \to A$. Define $g : J \to E$:
$g(1) = f(n_1)$ Where n_1 is the first number in $f^{-1}(E)$
$g(k) = f(n_k)$ Where n_k is the k^{th} number in $f^{-1}(E)$
g is well-defined, surjective, and injective

Thm $\mathbb{R}$ is uncountable

Pf Consider S defined as the set of infinite sequences of 0s and 1s. Every $s \in S$ can be taken to represent a real number, so $S \subset R$.
Let $s_i, i \in \mathbb{N}$ be a sequence of such sequences. Construct s to differ from each s_i in the i^{th} element. Clearly, $\forall i, s \neq s_i$.

Thm Let $\{E_n\}, n \in \mathbb{N}$ be a sequence of countable sets. Then $S = \bigcup_{n=1}^{\infty} E_n$ is countable.

Pf arrange E_n into sequence $x_{n,k}, k \in \mathbb{N}$. Arrange these in a table, and count along the diagonals.

Cor $\mathbb{Q}$ is countable.

Pf Define $E_n = \{\frac{m}{n} \forall m \in \mathbb{Z}\}$. $\mathbb{Q} = \bigcup_{n=1}^{\infty} E_n$ Therefore, $\mathbb{Q}$ is countable.