18.100B Lecture Notes

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<u>*Defn*</u> $\underline{\mathbb{C}}$ is the set $\mathbb{R}^2$ containing pairs (a, b) with operations:

$$(a, b) + (c, d) = (a + c, b + d)$$

$$(a, b) * (c, d) = (ac - bd, ad + bc)$$

We write (a, b) as $a + bi$, with $i = (0, 1)$.
Note $\mathbb{C}$ contains $\mathbb{R}$ as a subfield in the form $(a, 0)$. We will write $\text{Re}(a+ib) = a$, and $\text{Im}(a + ib) = b$

<u>*Defn*</u> The <u>complex conjugate</u> of $z = a + ib$ is $\overline{z} = a - ib$
The complex conjugate has the following properties:

a) $\overline{(z + w)} = \overline{z} + \overline{w}$

b) $\overline{(zw)} = \overline{z}\,\overline{w}$

c) $z + \overline{z} = 2\text{Re}(z)$ and $z - \overline{z} = 2\text{Im}(z)$

d) $z\overline{z}$ is real and positive (if $z \neq 0$)

<u>*Defn*</u> $|z| = \sqrt{z\overline{z}}$

<u>*Claim*</u>

a) $|z| > 0$ if $z \neq 0$, and 0 otherwise

b) $|z| = |\overline{z}|$

c) $|zw| = |z||w|$

d) $|\text{Re}z| \leq |z|$

e) $|z + w| \leq |z| + |w|$ ("Triangle inequality")

Defn The vector space over $\mathbb{C}(C^n)$ consists of n-tuples $\vec{z} = (z_1, z_2, ...z_n)$ $z_n \in \mathbb{C}$

$\vec{z} + \vec{w} = (z_1 + w_z, z_2 + w_2, ...z_n + w_n)$

if $\lambda \in \mathbb{Z}, \lambda\vec{z} = (\lambda z_1, \lambda z_2, ...\lambda z_n)$

Defn The Hermition inner product is

$$< \vec{z}, \vec{w} >= \sum_{k=1}^{n} z_k \overline{w_k}$$

Note $< \vec{z}, \vec{z} >= \sum_{k=1}^{n} z_k \overline{z_k} \in \mathbb{R}, > 0$ $\vec{z} \neq (0, 0, ...)$

Defn $||\vec{z}|| = \sqrt{< \vec{z}, \vec{z}}$

Claim

a) $< \vec{z}, \vec{w} >= \overline{(} < \vec{w}, \vec{z} >)$

b) $< \lambda\vec{z}, \vec{w} >= \lambda < \vec{z}, \vec{w} >$
$< \vec{z}, \lambda\vec{w} >= \overline{\lambda} < \vec{z}, \vec{w} >$

c) $< \vec{z} + \vec{w}, \vec{s} >=< \vec{z}, \vec{s} > + < \vec{w}, \vec{s} >$

Thm (Cauchy-Schwartz inequality)

$$| < \vec{z}, \vec{w} > | \leq ||\vec{z}|| \cdot ||\vec{w}||$$

Equivalently:

$$| < \vec{z}, \vec{w} > |^2 \leq < \vec{z}, \vec{z} >< \vec{w}, \vec{w} >$$

$$\sum_{k=1}^{n} z_k \overline{w_k} \leq (\sum_{k=1}^{n} z_k \overline{z_k})(\sum_{k=1}^{n} w_k \overline{w_k})$$

Pf

$$
\begin{aligned}
0 &\leq <\vec{z}-\lambda\vec{w},\vec{z}-\lambda\vec{w}> \\
&= <\vec{z},\vec{z}> -\lambda <\vec{w},\vec{z}> \\
&\quad -\overline{\lambda} <\vec{z},\vec{w}> -\lambda\overline{\lambda} <\vec{w},\vec{w}> \\
&\text{Set } \lambda = \tfrac{<\vec{z},\vec{w}>}{<\vec{w},\vec{w}>} \\
0 &\leq <\vec{z},\vec{z}> -\frac{<\vec{z},\vec{w}>}{<\vec{w},\vec{w}>} <\vec{w},\vec{z}> \\
&\quad -\frac{<\vec{w},\vec{z}>}{<\vec{w},\vec{w}>} <\vec{z},\vec{w}> + \frac{<\vec{z},\vec{w}><\vec{w},\vec{z}>}{<\vec{w},\vec{w}>} \\
<\vec{z},\vec{w}><\vec{w},\vec{z}> &\leq <\vec{z},\vec{z}><\vec{w},\vec{w}> \\
|<\vec{z},\vec{w}>^2| &\leq <\vec{z},\vec{z}><\vec{w},\vec{w}>
\end{aligned}
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