18.00B Lecture Notes

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Defn A metric space X is a set X with some distance function $d : X \times X \to [0, \infty)$ s.t.

a) $d(p,p) = 0$ and $d(p,q) > 0$ if $p \neq q$

b) $d(p,q) = d(q,p)$

c) $d(p,q) \leq d(p,r) + d(r,q)$ ("Triangle inequality")

Ex

$$X = \text{rooms in MIT}$$

$$d(p,q) = \text{shortest walking distance between their centers}$$

Ex

$$\mathbb{R}, d(x,y) = |x - y|$$

a) trivial

b) trivial

c)

$$\text{First,} |x+y| \leq |x| + |y|$$

$$d(x,y) = |x - y + z - z| \leq |x - z| + |z - y| = d(x,z) + d(z,y)$$

Ex

$$C^n,\ d(\vec{z},\vec{w}) = \|\vec{z}-\vec{w}\|$$

a) Straightforward from definition

b) Trivial

c) As above, first show $\|\vec{z}+\vec{w}\| \le \|\vec{z}\| + \|\vec{w}\|$

$$\begin{aligned}
\|\vec{z}+\vec{w}\|^2 &= <\vec{z}+\vec{w},\vec{z}+\vec{w}> \\
&= <\vec{z},\vec{z}> + <\vec{z},\vec{w}> + <\vec{w},\vec{z}> + <\vec{w},\vec{w}> \\
&= \|\vec{z}\|^2 + \|\vec{w}\|^2 + 2\mathrm{Re}(<\vec{z},\vec{w}>) \\
2\mathrm{Re}(<\vec{z},\vec{w}>) &\le 2\| <\vec{z},\vec{w}> \| \le 2\|\vec{z}\|\|\vec{w}\| \\
\|\vec{z}+\vec{w}\|^2 &\le \|\vec{z}\|^2 + \|\vec{w}\|^2 + 2\|\vec{z}\|\|\vec{w}\| \\
&\Rightarrow \|\vec{z}+\vec{w}\| \le \|\vec{z}\| + \|\vec{w}\|
\end{aligned}$$

$$d(\vec{z},\vec{w}) = \|\vec{z}-\vec{w}+\vec{s}-\vec{s}\| \le \|\vec{z}-\vec{s}\| + \|\vec{s}-vw\| = d(\vec{z},\vec{s}) + d(\vec{w},\vec{s})$$

Ex Euclidean space: $R^n, d(\vec{x},\vec{y}) = ||\vec{x}-\vec{y}||$

$$\left(||\vec{x}|| = \sqrt{<\vec{x},\vec{x}>}\right)$$

Everything we said above about complex vector spaces still holds, since $\mathbb{R}$ is a subfield of $\mathbb{C}$

Claim if $E \subset X$, and (X,d) is a metric space, so is (E,d)

Ex R^2 with the "taxi driver" metric $d(\vec{x},\vec{y}) = |x_1-y_1| + |x_2-y_2|$

Ex X = a set of bounded functions on R, with $d(f,g) = \sup_{x\in\mathbb{R}} |f(x)-g(x)|$

Defn Let (X,d) be a metric space, and $p \in X$. The open ball $B_r(p)$ of radius $r > 0$ around p is

$$B_r(p) = \{x \in X \text{ s.t. } d(x,p) < r\}$$

We call the ball "open" because it doesn't contain its boundary (We use $< r$ instead of $\le r$)

Defn E ⊂ X is open (in the topology given by a metric d) if

$$\forall p \in E \; \exists \; r > 0 \text{ s.t. } B_r(p) \subset E$$

Lemma Open balls are open.

Pf Given $q \in B_r(p)$, we want to find some $B_\epsilon(q)$ contained in X

$$\begin{aligned} q \in B_r(p) &\Leftrightarrow d(p,q) < r \\ \text{Set } \epsilon &= \frac{r - d(p,q)}{2} \\ x \in B_\epsilon(q) &\Rightarrow d(p,x) \leq d(p,q) + d(q,x) \\ &\leq d(p,q) + \epsilon < r \\ &\Rightarrow x \in B_r(p) \end{aligned}$$

Defn A limit point $p \in X$ of a set $E \subset X$ satisfies

$\forall r > 0 \exists q \neq p \subset (B_r(p) \bigcap E)$