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18.510 Problem Set

Projects 31 and 32

1. Project 31. THEOREM 1.1. *Each $n \in N$ is Dedekind finite.*

Proof. For $n = 0$, n is Dedekind finite since n has no proper subsets.

For $n = 1$, n is Dedekind finite since the only proper subset of 1 is $\emptyset$, and there is no function from n to $\emptyset$.

The rest will be done by induction. Suppose we know that n is Dedekind finite. We wish to prove that $S(n)$ is Dedekind finite. Suppose $S(n)$ is Dedekind infinite. Then there is a one-to-one function from $S(n)$ onto a proper subset of $S(n)$. Call this function f. Either n is in the image of f or it is not.

Suppose n is not in the image of f. Then the image of f is then a subset of (and possibly equal to) n. Let $a = f(n)$. Now f restricted to n is a function from n to a subset of n. Since f is one-to-one, a has only one pre-image, i.e. n. This means f restricted to n is a function from n to a subset of $n\backslash\{a\}$, which is a proper subset of n. We now have a function which maps n one-to-one into a proper subset of itself, showing that n is Dedekind infinite. But by assumption, n is Dedekind finite.

Supposel, on the other hand, that n is in the image of f. Let b be the unique element of $S(n)$ for which $f(b) = n$. Then define a function g from $S(n)$ into $S(n)$ as follows: let $g(b) = f(n)$ and let $g(n) = f(b) = n$, and let $g(x) = f(x)$ for all other values of $x \in S(n)$. Since f is one-to-one, g is also one-to-one. Also, g restricted to n maps into n, since $g(n) = n$ and g is one-to-one. Furthermore, since f does not map onto $S(n)$, neither does g, and since n is in the image of g, g restricted to n does not map onto n. We now have a function which maps n one-to-one into a proper subset of itself, showing that n is Dedekind infinite. But by assumption, n is Dedekind finite. □

THEOREM 1.2. *If X is finite then it is Dedekind finite.*

Proof. If X is finite, then there is a function h that maps X one-to-one and onto n, for some $n \in N$. Now suppose X were Dedekind infinite. Then there would be a function g that maps X one-to-one into a proper subset of itself. Then $h \circ g \circ h^{-1}$ maps n one-to-one into n. Furthermore, since for some b, $g(a) = b$ for no a, then $h \circ g \circ h^{-1}(a) = h(b)$ for no a, so we have a function that maps n one-to-one into a proper subset of itself. But we have proven that n is Dedekind finite. So X must be Dedekind finite. □

THEOREM 1.3. *If X is infinite then there is a function mapping ω one-to-one into X.*

Proof. By the well-ordering theorem, we can well-order X and derive the successor function $S'(a)$ as the least element of the set of all elements of X greater than a. The function is not always defined, because there may be some element b which is larger than all other elements.

We will now recursively define a map $h : \omega \to X$. First, $h(0)$ is the least element of X. Then, $h(S(n)) = S'(h(n))$.

If $S'(h(n))$ is not defined, then $h(n)$ must be the largest element, and h defined up to n (or rather, restricted to $S(n)$), constitutes a one-to-one map from $S(n)$ onto X, contradicting the statement that X was infinite. Therefore, this procedure defines an $h(n)$ for all $n \in \omega$, in such a way that h is which is one-to-one. □

THEOREM 1.4. *If X is infinite then it is Dedekind infinite.*

Proof. Use the same successor function $S' : X \to X$ and $h : \omega \to X$ in the previous proof, and let Y be the image of h. Then define a map $g : X \to X$ so that $g(x) = S'(x)$ for $x \in Y$, and $g(x) = x$ for $x \notin Y$. Then g is a one-to-one function from X into $X\backslash\{$the smallest element of X$\}$, which is a proper subset of X. Therefore, X is Dedekind infinite. □

2. Project 32. THEOREM 2.1. *The countable union of countable sets is countable.*

Proof. Let A be a countable set of countable sets A_i. This means there is a one-to-one function $f : A \to \omega$, and a family of one-to-one functions $f_i : A_i \to \omega$. We wish to construct a one-to-one function $g : \bigcup A_i \to \omega$.

First, we define recursively a one-to-one function $k : N \to N \times N$ as follows:

$k(0) =< 0, 0 >$.

If $k(a) =< m, 0 >$, then $k(S(a)) =< 0, S(m) >$.

If $k(a) =< m, S(n) >$, then $k(S(a)) =< S(m), n >$.

To see that k is one-to-one, note that $m + n$ never decreases, and for a given level of $m + n$, n always increases.

We also define $h : N \times N \to N$ so that $h(< m, n >) = f_m^{-1}(n)$, and let $Y \subseteq N \times N$ be the preimage of h, so that Y is the set of all pairs of natural numbers for which $f_m^{-1}(n)$ is defined. Note that Y is well-ordered (furthermore, enumerated) by h^{-1}. We can define a successor function S' as before.

Now we define g.

First, $g(0) = h(b)$, where b is the smallest element of Y.

If $g(a) = h(c)$, then $g(S(a)) = h(S'(c))$ □

THEOREM 2.2. *If A is countable, then $A \times A$ is countable.*

Proof. Let $B_m \subset A \times A$ be the set of ordered pairs $< m, n >$ (B is indexed by A).

Let B be the set of all B_i.

B is a countable set of countable sets. Its union is the collection of all ordered pairs of A, that is, $A \times A$. By the previous theorem, $\bigcup B_i = A \times A$ is countable. □

THEOREM 2.3. *There is no largest cardinal.*

Proof. Suppose there were a largest cardinal κ. Then $||\mathcal{P}(\kappa)|| \leq ||\kappa||$. But we already know that $||\kappa|| < ||\mathcal{P}(\kappa)||$ by Cantor's diagonalization procedure. This contradicts our assumption. □