6.972 Optical Networks 19 September 2001

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Problem set #1

Problem 1.1 (OOK signalling with noise): Consider an OOK signalling optical transmission scheme in which the bit period is T_o and the equally likely laser modulating signal waveforms are

$$\begin{aligned} s_0(t) &= 0, && 0 \le t \le T_o, \\ s_1(t) &= \sqrt{\frac{\mathcal{E}}{T_o}}, && 0 \le t \le T_o. \end{aligned}$$

Assuming direct detection at the receiver and that photon arrivals form a Poisson process, let λ_o and $\alpha\lambda_o$ ($0 < \alpha < 1$) be the photon arrival rates when the modulating signal is nonzero and zero respectively. (The parameter α is refered to as the extinction ratio.) More specifically, given that $s_i(t)$ was transmitted and neglecting transmission delay, the photon arrival rate $\lambda(t)$ is given by

$$\begin{aligned} \lambda(t) &= \alpha\lambda_o, \quad 0 \le t \le T_o, && \text{for } i = 0, \\ \lambda(t) &= \lambda_o, \quad 0 \le t \le T_o, && \text{for } i = 1. \end{aligned}$$

The detection statistics $\mathbf{t}$ indicates the number of photon arrivals n as well as their arrival times $(t_1, ..., t_n)$ during the bit period.

(a) Derive the optimal decision rule for the receiver which minimizes the probability of bit error. Argue that photon counting is optimal. (In other words, the number of photon arrivals in the bit period serves as a sufficient statistics for the decision problem.) More specifically, express the decision rule in the form

$$n \quad \begin{matrix} \Omega_1 \\ > \\ \le \\ \Omega_0 \end{matrix} \quad \gamma.$$

You can assume and use the PDF of $\mathbf{t}$ below.

$$p(\mathbf{t}|\lambda(t)) = \left[\prod_{i=1}^{n} \lambda(t_i)\right] e^{-\int_0^{T_o} \lambda(\eta)d\eta}$$

(b) Give the exact expression for the probability of bit error P_b.

(For further analysis, you may use the Chernoff bound to get an upperbound expression for P_b above. We shall not do so here. Problem 1.3 will ask you to apply the Chernoff bound.)

Problem 1.2 (M-ary hypothesis testing): Consider the decision problem with M hypotheses $H_0, ..., H_{M-1}$. Let $p_0, ..., p_{M-1}$ denote their prior probabilities. Given an observation y, show that the optimal decision rule which minimizes the proability of decision error is to decide on H_i such that

$$i = \arg \max_{0 \le j \le M-1} \; p_j \; p(y|H_j).^1$$

(Equivalently, in terms of likelihood ratios $\Lambda_{i,j}(y)$, we decide on H_i such that

$$\Lambda_{i,j}(y) = \frac{p(y|H_i)}{p(y|H_j)} \ge \frac{p_j}{p_i}, \quad \text{for all } j \ne i.)$$

Problem 1.3 (M-ary PPM): Consider an M-ary PPM optical transmission scheme in which the symbol period is T_p and the equally likely laser modulating signal waveforms are

$$s_i(t) = \begin{cases} \sqrt{\frac{M\mathcal{E}}{T_p}}, & i\frac{T_p}{M} < t \le (i+1)\frac{T_p}{M}, \\ 0, & \text{otherwise,} \end{cases}$$

for $0 \le i \le M-1$.

Assuming direct detection at the receiver and that photon arrivals form a nonhomogeneous Poisson process, let λ_p and $\alpha\lambda_p$ ($0 < \alpha < 1$) be the photon arrival rates when the modulating signal is nonzero and zero respectively. More specifically, given that $s_i(t)$ was transmitted, the photon arrival rate $\lambda(t)$ is given by

$$\lambda(t) = \begin{cases} \lambda_p, & i\frac{T_p}{M} < t \le (i+1)\frac{T_p}{M}, \\ \alpha\lambda_p, & \text{otherwise.} \end{cases}$$

[1] $\arg\max_j f(j)$ is an index i such that $f(i)$ is the maximum, i.e. $f(i) \ge f(j)$ for all $j \ne i$.

The detection statistics $\mathbf{t}$ indicates the number of photon arrivals n as well as their arrival times $(t_1, ..., t_n)$ during the symbol period.

(a) Derive the optimal decision rule for the receiver which minimizes the probability of symbol error. Argue that photon counting in each of the M intervals, $(i\frac{T_p}{M}, (i+1)\frac{T_p}{M}]$ for $0 \leq i \leq M-1$, is optimal.

(b) Use the union bound and the Chernoff bound to derive the upper bound on the probability of symbol error given below.

$$P_s \leq (M-1)e^{-\lambda_p \frac{T_p}{M}(1-\sqrt{\alpha})^2}$$

Problem 1.4 (Noiselss OOK versus noiseless M-ary PPM): Consider the noiselss OOK system and noiselss M-ary PPM system in problems 1.1 and 1.3 ($\alpha = 0$).

For noiseless OOK, we know from class (lectures 3-4) that the optimal decision rule is to decide H_0 if there is no photon arrival in the bit period and to decide H_1 otherwise. The corresponding probability of bit error P_b is $\frac{1}{2}e^{-\lambda_o T_o}$.

(a) For noiseless M-ary PPM, derive the optimal decision rule, and show that the probability of symbol error P_s is at most $e^{-\lambda_p T_p / M}$.

(b) Define the receiver efficiency as the expected number of transmitted information bits per photon arrival for a fixed probability of bit error P_b. For the purpose of this problem, assume that $P_b = \frac{1}{2}e^{-\theta}$ for some $\theta > 0$.

For M-ary PPM, the probability of having a particular error bit P_b is at most P_s since a symbol error may not result in the bit error. A conventional approximation is to consider $P_b \approx \frac{1}{2}P_s$, i.e. given that a symbol is in error, the particular bit is in error about half of the time. Using a conservative approximation $P_s \approx e^{-\lambda_p T_p / M}$ for M-ary PPM and $P_b \approx \frac{1}{2}e^{-\lambda_o T_o}$ for OOK, approximate and compare the receiver efficiency in the two cases. What is the receiver efficiency for noiseless M-ary PPM as M approaches infinity?

(c) The sampling theorem tells us that a band-limited signal with duration T and bandwidth W possesses $2WT$ degrees of freedom. For noiseless M-ary PPM, the modulating signals $s_i(t)$ possess at least M degrees of freedom.

For a fixed bit rate, what happens to the bandwidth of modulating signals $s_i(t)$ as M approaches infinity? Alternatively, for a fixed bandwidth of modulating signals $s_i(t)$, what happens to the bit rate as M approaches infinity?

Problem 1.5 (OPTIONAL): Derive the PDF of the observed statistics $\mathbf{t}$ given in problem 1.1(a) from the following definition of a nonhomogeneous Poisson process $\{N(t); t \geq 0\}$:

1. $N(0) = 0$.
2. The process has statistically independent increments.
3. The probability of arrivals is described as

$$\begin{aligned} P(N(t+\Delta t) - N(t) = 1) &= \lambda(t)\Delta t, \\ P(N(t+\Delta t) - N(t) > 1) &= o(\Delta t), \end{aligned}$$

   where $\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0$.