6.972 Optical Networks — 1 October 2001

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Problem set #2

Problem 2.1 (Implementation of photon counting receivers): Consider an M-ary transmission scheme (e.g. M-ary PPM in problem 1.3) with equally likely hypotheses $H_0, ..., H_{M-1}$ and the symbol period T. Under hypothesis H_i, $0 \le i \le M-1$, the photon arrival rate is $\lambda_i(t), 0 \le t \le T$. Assume a steady background noise so that $\lambda_i(t)$ is nonzero at all time.

The detection statistics $\mathbf{t}$ indicates the number of photon arrivals n as well as their arrival times $(t_1, ..., t_n)$ during the symbol period. In addition, the received signal can be expressed as a sequence of arrival impulses (q denotes the size of an electron charge)

$$r(t) \;=\; q \sum_{k=1}^{n} \delta(t - t_k).$$

(a) Figure 2.1 shows the schematic diagram of a *correlation* receiver. Find the expressions of $g_i(t)$ and γ_i, $0 \le i \le M-1$, for the optimal receiver that minimizes the probabilty of symbol error.

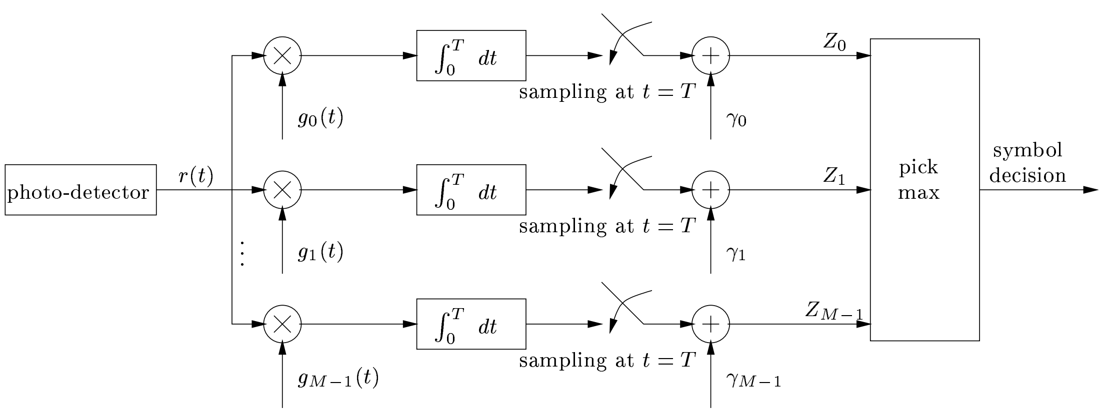

Figure 2.1: Correlation receiver

(As in problem 1.1a, given the photon arrival rate $\lambda_i(t)$, the PDF of $\mathbf{t}$ is

$$p(\mathbf{t}|\lambda_i(t)) = \left[\prod_{k=1}^{n} \lambda_i(t_k)\right] e^{-\int_0^T \lambda_i(\eta) d\eta}.)$$

(b) Figure 2.2 shows the schematic diagram of a *matched filter* receiver. Find the expressions of $h_i(t)$ and ξ_i, $0 \leq i \leq M-1$, for the optimal receiver that minimizes the probabilty of symbol error.

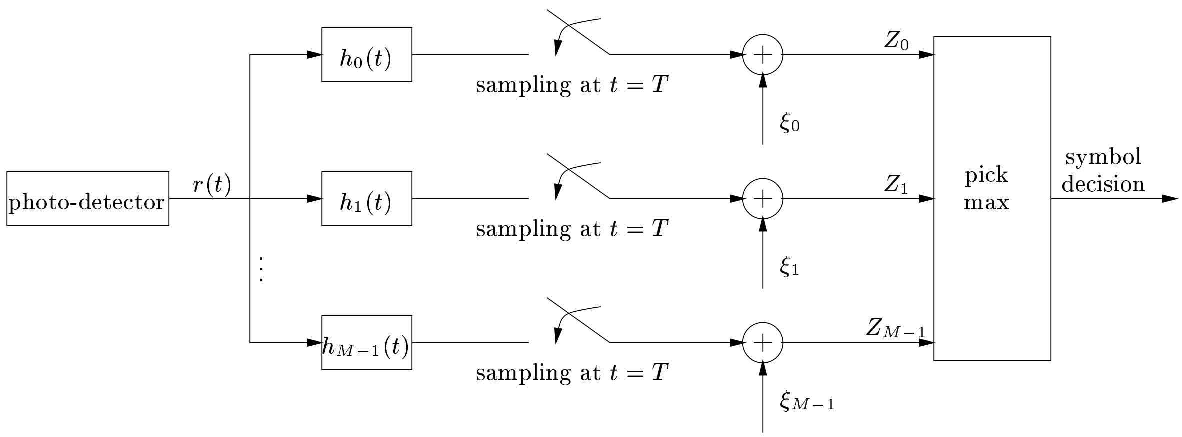

Figure 2.2: Matched filter receiver

Problem 2.2 (Binary hypothesis testing with Gaussian random variables): Consider a binary transmission scheme with two hypotheses H_0 and H_1 (bit 1 and bit 0) and the bit period T_o.

In detection analysis, given sufficient light intensity, a filtered Poisson point process is approximated reasonably well by a Gaussian random process.[1] After optical detection, electronic amplification, and electronic processing at the receiver (e.g. matched filter and sampling), the observation can be described as a Gaussian random variable Y. Assume its mean and variance are

$$
\begin{aligned}
E[Y] &= m_0,\ Var[Y] = \sigma_0^2, && \text{under } H_0,\\
E[Y] &= m_1,\ Var[Y] = \sigma_1^2, && \text{under } H_1.
\end{aligned}
$$

(a) Do problem 4.6 in the book [RS98,p.198].

(b) Do problem 4.5 in the book [RS98,p.198]. (Use the approximate expression for the decision threshold T_d under high SNR in part (a). Note that in this problem, $m_0 = I_0$ and $m_1 = I_1$.)

[1] Note that a Poisson point process has the form $q\sum_i \delta(t-t_i)$, while a Poisson counting process has the staircase form as shown in lectures 3-4.

Problem 2.3 (Chernoff bound for Gaussian random variables): Using the Chernoff bound, show that for a zero-mean Gaussian random variable X with variance σ^2,

$$P(X > x) \;\leq\; \exp\left[-\frac{x^2}{2\sigma^2}\right].$$

Problem 2.4 (Error exponents for homodyne detection): Figure 2.3 shows a schematic diagram for homodyne detection. Let $s(t)$ denote the data signal which modulates the transmit laser. Assuming no transmission loss, the square-law detector output is given by

$$|s(t)|^2 + |l|^2 + 2Re(l^* s(t)),$$

where l is the amplitude of the local oscillator signal. For simplicity, assume l is real. Assume also that $|l|^2$ is much larger than $|s(t)|^2$. Consequently, we can neglect the term $|s(t)|^2$, and consider that all shot noise comes from the term $|l|^2$.

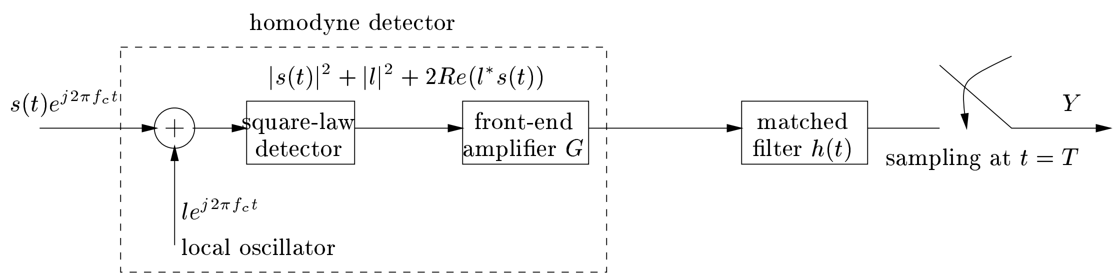

Figure 2.3: Homodyne detection

Let $\mathcal{R}$ denote the *responsivity* of the photo detecter. A light signal arriving at power P is converted to a stream of photon arrivals of with rate $\mathcal{R}P/q$ (a current of $\mathcal{R}P$), where q denotes the size of an electron charge.

The square-law detector output can be modeled as a Poisson point process. The front-end electronic amplifier is a wide-band low pass filter with current gain G (G is real and positive) and the pass band much greater than the data signal bandwidth. The output of this front-end amplifier is a filtered Poisson process which is well approximated by a Gaussian random process.

The expected output current from the homodyne detector is $2\mathcal{R}Gls(t) + \mathcal{R}Gl^2$. The dominant noise is the shot noise that comes from the term $\mathcal{R}Gl^2$, and has the two-sided power spectral density, denoted by $N_0/2$, equal to a constant $q\mathcal{R}G^2l^2$ in the amplifier pass band.

For analytical purpose, we can consider the detector output current as the sum of the signal $\tilde{s}(t) = 2\mathcal{R}Gls(t) + \mathcal{R}Gl^2$ and the zero-mean additive white Gaussian noise (AWGN) process $\{N(t)\}$ with power spectral density $q\mathcal{R}G^2l^2$. (Although the noise is clearly band-limited by the front-end amplifier, since the amplifier bandwidth is much greater than the matched filter bandwidth, we can reasonably approximate the noise as being white.)

(a) **On-Off Keying (OOK):** In practice, for OOK, the signal modulating waveform is not a rectangular pulse as given in problem 1.1 since the associated bandwidth would be infinite. Instead, the ON modulating signal $s_1(t)$ is usually selected to be band-limited to the bandwidth of $\approx 1/(2T)$ where T is the bit period.

Consider the ideal case where the signal bandwidth is exactly $1/(2T)$, and $s_1(t)$ is given by

$$s_1(t) \;=\; \sqrt{\frac{\mathcal{E}}{T}}\; sinc\left(\frac{t}{T}\right),$$

where $sinc(x) \;=\; \frac{\sin(\pi x)}{\pi x}$. In this case, the matched filter $h(t)$ is an ideal low pass filter with bandwidth $1/(2T)$.

Construct the optimal decision rule based on the observation Y as indicated in figure 2.3. Find the expression for the bit error probability P_b, and use the Chernoff bound to show that

$$P_b \;\leq\; \exp\left[-N_{\mathrm{OOK}}\right],$$

where N_{OOK} is the expected number of photon arrivals in each bit period due to the data signal $s(t)$, i.e. $N_{\mathrm{OOK}} = \frac{\mathcal{R}\mathcal{E}}{2q}$.

(b) **Phase Shift Keying (PSK):** For PSK, instead of $s_0(t) = 0$, we assign

$$s_0(t) \;=\; -s_1(t) \;=\; -\sqrt{\frac{\mathcal{E}}{T}}\; sinc\left(\frac{t}{T}\right).$$

Using the same receiver structure, show that the bit error probability P_b is now bounded by

$$P_b \;\leq\; \exp\left[-2N_{\mathrm{PSK}}\right],$$

where N_{PSK} is the expected number of photon arrivals in each bit period due to the data signal $s(t)$, i.e. $N_{\mathrm{PSK}} = \frac{\mathcal{R}\mathcal{E}}{q}$.