Geoffrey Thomas
18.781 pset 10
Collaborators: Liz Denys

5.3 3. For the arithmetic progression, since $r^2 - s^2 < 2rs$, we set the constant difference to s^2, so we have $2rs = r^2$, or $2s = r$. Plugging back into the original equation, we have the triple $(3s^2, 4s^2, 5s^2)$ for all integers s, and all multiples of it, i.e., all multiples of $(3, 4, 5)$ and no other triples.

As far as geometric progressions, any such progression would need to satisfy $x^2 + ax^2 = a^2x^2$, which is only true if $1 + a = a^2$, which has no solutions in the integers. (There are real-valued Pythagorean triples whose sides form a geometric progression.)

5.3 4. Write $u = gm$ and $v = gn$, so that $(m, n) = 1$. Then $uv = g^2mn$ is a perfect square, so mn must also be a perfect square (since it is the quotient of two perfect squares, and an integer). So by lemma 5.4, m and n are also perfect squares r^2 and s^2, fitting the original form.

5.3 7. Note that we can rewrite the equation in the form $(x + y)(x - y) = n$. For any odd integer $n = 2m + 1$, we can set $x = m + 1$ and $y = m$, giving the representation $(2m + 1)(1)$. For any multiple of 4 $n = 4k$, we can set $x = k + 1$ and $y = k - 1$ to give the representation $(2k)(2)$. For $n \equiv 2 \pmod 4$, since the only quadratic residue modulo 4 is 1, x^2 and y^2 can only be 0 or 1, and so it is impossible to have any x and y such that $x^2 - y^2 \equiv 2 \pmod 4$.

5.3 8. Pythagorean triples take the form $(r^2 - s^2, 2rs, r^2 + s^2)$ for integers $r > s > 0$. From the previous problem, if $n \not\equiv 2 \pmod 4$, then we can find some (r, s) such that $n = r^2 - s^2$, the first element in a triple. If $n \equiv 2 \pmod 4$, then let $r = \frac{n}{2}$ and $s = 1$, so that $n = 2rs$, the second element in a triple.

5.3 12. As desired,

$$(\pm(r^2 - 5s^2))^2 + 5(2rs)^2 = (r^2 + 5s^2)^2$$
$$r^4 - 10r^2s^2 + 25s^2 + 20r^2s^2 = r^4 + 10r^2s^2 + 25s^2$$

Solving for $(x, y, z) = (2, 3, 7)$, assuming the positive branch for x, we have $x + z = 9 = 2r^2$, which has no rational solution for r. Assuming the negative branch, $x + z = 9 = 10s^2$, which has no rational solution for s. (If we attempted to switch x and y, then we'd have $y = 2 = 2rs$, so $r = \frac{1}{s}$. $z = 7 = r^2 + 5s^2 = s^{-2} + 5s^2$ then has no rational solution for s.)

5.3 15. Let $r = s + d$. Then the two legs are $(s + d)^2 - s^2 = 2sd + d^2$ and $2(s + d)s = 2s^2 + 2sd$. For these two to be exactly equal, we would need a solution to $d^2 = 2s^2$ over the integers, which does not exist, so we cannot have the les be exactly equal. However, we can get the legs arbitrary close to equal compared to the magnitudes of d and s, for instance by setting $s = 10^i$ and $d = \left[\sqrt{2}10^i\right]$.

5.4 1. This is a special case of $x^2 + y^2 \equiv 3 \pmod 9$. The quadratic residues modulo 9 are 1, 4, and 7; no sum of two of those, modulo 9, gives 3.

5.4 4. Compare the highest power of two in both sides of $a^2 + b^2 + c^2 = 2abc$. It cannot be 2^1, since that would imply that 2 does not divide abc, which would make the left side odd. If it is

2^2, then exactly one of x, y, and z is even. This means the left side, modulo 4, is equivalent to $4+1+1 \equiv 2$, but the right side is congruent to 4, which also does not work. If it is 2^3 or more and we make two variables evenn, then the left side is odd. If it is 2^4 or more and three variables are even, then we have a smaller solution $x/2, y/2, z/2$ by Fermat's method od descent, so the only minimal solution is $0, 0, 0$.

5.4 5. Consider this equation modulo 9. We want the sum of two quadratic residues to equal three times the sum of two quadratic residues. The quadratic residues modulo 9 are 1, 4, and 7; to this list add zero. The possible sums of two of those are 0, 1, 2, 4, 5, 7, 8 modulo 9. Three times those residues are 0, 3, and 6. So, the only way for the two sides to be congruent modulo 9 is if they are each zero. This specifically means that each square must be congruent to 0 modulo 9, so we have $(3a)^2 + (3b)^2 = 3((3c)^2 + (3d)^2)$ for some (a, b, c, d), which reduces to $a^2 + b^2 = 3(c^2 + d^2)$. But this is a smaller solution than any possible original solution (x, y, u, v). So, by Fermat's method of descent, any purported minimal solution would imply a smaller solution $(x/3, y/3, u/3, v/3)$, which yields a contradiction.

5.4 16. Consider a minimal Pythagorean triple (a, b, c). We know (as a general property of Pythagorean triples) that one of a or b is even and one is odd; without loss of generality let b be even and write it as $2z$. The area of the triangle is then za, which we want to be a square Since $(z, a) = 1$, this implies both z and a are squares. We can then write the triple as $(u^2, 2v^2, c)$. From theorem 5.5, though, we have $u^2 = r^2 - s^2$, or $u^2 + s^2 = r^2$. The terms in this equation are all strictly lower than the corresponding terms in the original equation $a^2 + b^2 = c^2$, so by Fermat's method of descent, we have a contradiction to the claim that our original triple was minimal. Therefore there are no solutions.