Soft-Collinear Factorization and Zero-Bin Subtractions

Aneesh Manohar

University of California, San Diego

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Outline

Chiu, Golf, Kelley, AM, PRL 100 (2008) 021802

Chiu, Golf, Kelley, AM, PRD 77 (2008) 053004

Chiu, Kelley, AM, PRD 78 (2008) 073006

Chiu, Fuhrer, Hoang, Kelley, AM, PRD 79 (2009) 053007

Talked about the first three at SCET08 in Mainz

Toy Theory

Consider a $SU(2)$ gauge theory (completely) spontaneously broken by a Higgs in the fundamental representation.

All gauge bosons have a common mass M.

Infrared structure perturbative and regulated by M

The standard model omitting the $U(1)$ sector. (only $W^\pm$, W^3)

Write group theory in terms of C_F, C_A — only makes sense for $SU(2)$
Otherwise $SU(N) \to SU(N-1)$ and some gauge bosons remain massless.

Fermions have a mass m

Large scale Q, with $Q \gg M$, $Q \gg m$, and neglect M^2/Q^2 and m^2/Q^2 power corrections.

Notation

$$L_Q = \log \frac{Q^2}{\mu^2}$$

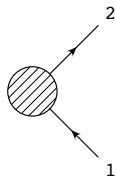$$L_M = \log \frac{M^2}{\mu^2}$$

$$L_{Q/M} = \log \frac{Q^2}{M^2} = L_Q - L_M$$

Sudakov Form Factor

Sufficient to compute *n*-particle hard scattering.

Linear in $\log p_i \cdot p_j$ to all orders   Chiu et al. (2008)

Consider the scattering of a fermion by an external current $\bar{\psi}\gamma^\mu\psi$

$$Q^2 = -q^2 = 2p_1 \cdot p_2$$

$$F_E(Q^2/M^2)\ [\bar{u}(p_2)\gamma^\mu u(p_1)] = \langle p_2|\bar{\psi}\gamma^\mu\psi|p_1\rangle$$

Contains $\alpha \log^2(Q^2/M^2)$ — Sudakov double logarithms

Structure of Terms

$$
\mathcal{S} = \begin{pmatrix}
1 \\
\alpha\mathsf{L}^2 & \alpha\mathsf{L} & \alpha \\
\alpha^2\mathsf{L}^4 & \alpha^2\mathsf{L}^3 & \alpha^2\mathsf{L}^2 & \alpha^2\mathsf{L} & \alpha^2 \\
\alpha^3\mathsf{L}^6 & & \cdots \\
\vdots
\end{pmatrix}
$$

In the leading-log regime $\mathsf{L} \sim 1/\alpha$, the various terms are of order

$$
\mathcal{S} = \begin{pmatrix}
1 \\
\frac{1}{\alpha} & 1 & \alpha \\
\frac{1}{\alpha^2} & \frac{1}{\alpha} & 1 & \alpha & \alpha^2 \\
\frac{1}{\alpha^3} & & \cdots \\
\vdots
\end{pmatrix}.
$$

Infrared Evolution Equation

Collins formula for the Sudakov form factor

$$\log F_E(Q^2) = \log F_0(M) + \int_{M^2}^{Q^2} \frac{\mathrm{d}\mu^2}{\mu^2} \left[\zeta(\mu) + \xi(M) + \int_{M^2}^{\mu^2} \frac{\mathrm{d}\mu'^2}{\mu'^2} \Gamma(\mu') \right]$$

F_0, ζ, ξ and Γ are functions only of α at the relevant scale.

Γ is the cusp anomalous dimension.

ξ integral can be done to give

$$\xi\left(\alpha\left(M\right)\right) \, \log Q^2/M^2$$

Used by the Karlsruhe group for electroweak corrections.
Compare with fixed order computation.

Kuhn, Jantzen, Penin, Smirnov

SCET Formula

$$\log F_E(Q^2) = C(\alpha(Q))$$

$$+ \int_Q^M \frac{\mathrm{d}\mu}{\mu} \left[A(\alpha(\mu)) \log \frac{\mu^2}{Q^2} + B(\alpha(\mu)) \right]$$

$$+ D_0(\alpha(M)) + D_1(\alpha(M)) \log \frac{Q^2}{M^2}$$

- C: matching at Q — high scale matching
- $A \log \mu^2/Q^2 + B$: SCET anomalous dimension
- $D_0 + D_1 \log Q^2/M^2$: matching at M — low scale matching
- There is a $\log Q$ in the matching at M
- Equivalent to Infrared Evolution Equation

Resummation: Exponentiated Form

Exponentiated form:

$$\log \mathcal{S} \;=\; \begin{pmatrix} \alpha \mathsf{L}^2 & \alpha \mathsf{L} & \alpha & & \\ \alpha^2 \mathsf{L}^3 & \alpha^2 \mathsf{L}^2 & \alpha^2 \mathsf{L} & \alpha^2 & \\ \alpha^3 \mathsf{L}^4 & \alpha^3 \mathsf{L}^3 & \alpha^3 \mathsf{L}^2 & \alpha^3 \mathsf{L} & \alpha^3 \\ \alpha^4 \mathsf{L}^5 & & & \cdots & \\ \vdots & & & & \end{pmatrix}$$

In the leading-log regime:

$$\log \mathcal{S} \;=\; \begin{pmatrix} \frac{1}{\alpha} & 1 & \alpha & & \\ \frac{1}{\alpha} & 1 & \alpha & \alpha^2 & \\ \frac{1}{\alpha} & 1 & \alpha & \alpha^2 & \alpha^3 \\ \frac{1}{\alpha} & & \cdots & & \\ \vdots & & & & \end{pmatrix}.$$

Resummation: Exponentiated Form

$$\log \mathcal{S} = \frac{1}{\alpha} f_0 + f_1 + \alpha f_2 + \ldots = \frac{1}{\alpha} \left[f_0 + \alpha f_1 + \alpha^2 f_2 + \ldots \right]$$

so that f_1 and f_2 are corrections to $\log A$. However,

$$\mathcal{S} = \exp \left[\frac{1}{\alpha} f_0 + f_1 + \alpha f_2 + \ldots \right] = e^{\frac{1}{\alpha} f_0} \times e^{f_1} \times e^{\alpha f_2} \times \ldots$$

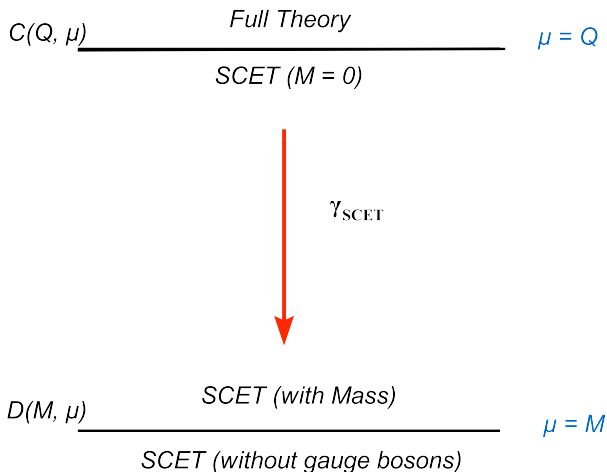Must include the LL and NLL series. The NLL series is *not* a correction.

LL: one-loop cusp
NLL: two-loop cusp, one-loop non-cusp, one-loop D_1
NNLL: three-loop cusp, two-loop non-cusp and D_1, one-loop C and D_0
Literature: One-loop LL, NLL, etc.

Outline of Calculation

$C(Q, \mu)$ _____ *Full Theory* _____ $\mu = Q$

SCET (M = 0)

γ_{SCET}

$D(M, \mu)$ _____ *SCET (with Mass)* _____ $\mu = M$

SCET (without gauge bosons)

Sudakov Form Factor at One Loop

$$F_E(Q^2) = 1 + \frac{\alpha\, C_F}{4\pi}\left[-\mathsf{L}_{Q/M}^2 + 3\mathsf{L}_{Q/M} - \frac{7}{2} - \frac{2\pi^2}{3}\right]$$

High Scale Matching:

$$C(Q,\mu) = 1 + \frac{\alpha\, C_F}{4\pi}\left[-\mathsf{L}_Q^2 + 3\mathsf{L}_Q + \frac{\pi^2}{6} - 8\right]$$

Running:

$$\gamma(\mu) = \frac{\alpha\, C_F}{4\pi}\left[4\mathsf{L}_Q - 6\right]$$

Low Scale Matching:

$$D(Q,M,\mu) = 1 + \frac{\alpha\, C_F}{4\pi}\left[2\mathsf{L}_M\mathsf{L}_Q - \mathsf{L}_M^2 - 3\mathsf{L}_M + \frac{9}{2} - \frac{5\pi^2}{6}\right]$$
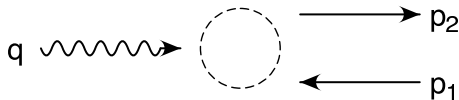
Factorization of Scales

EFT has factored the Q and M dependence.

The Q dependence goes into the matching condition, plus the single-log D_1 term in the low-scale matching in the EFT.

The M dependence is in the EFT computation.

Breit Frame

Work in the Breit frame where the two particles are back to back,

and move in the directions given by the null vectors

$$n^\mu = (1, \mathbf{n}), \quad \bar{n}^\mu = (1, -\mathbf{n})$$

and define

$$p^+ = n \cdot p, \quad p^- = \bar{n} \cdot p, \quad p^\mu = \frac{1}{2} n^\mu (\bar{n} \cdot p) + \frac{1}{2} \bar{n}^\mu (n \cdot p) + p_\perp^\mu$$

EFT Modes

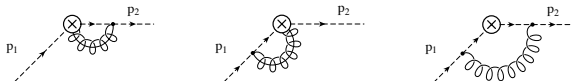EFT has *n*-collinear $\bar{n}$-collinear, and soft mass modes:

$$
\begin{array}{cccc}
& p^- & p^+ & p_\perp \\
n\text{-collinear} : & Q & Q^2/M & M \\
\bar{n}\text{-collinear} : & Q^2/M & Q & M \\
\text{mass} : & M & M & M
\end{array}
$$

Introduce fields $\xi_{n,p}$, $\xi_{\bar{n},p}$, $A_{n,p}$, $A_{\bar{n},p}$, A. Collinear gauge invariance requires fields in the combination

$$\left[W_n^\dagger \xi_n \right]$$

W_n is a collinear Wilson line in the $\bar{n}$ direction, and contains A_n.

Graphs in SCET

Would like to factor the EFT amplitude into three pieces.
The n-collinear diagram: ($f_\epsilon = (4\pi)^{-\epsilon} \mu^{2\epsilon} e^{\epsilon \gamma_E}$)

$$I_n = -2ig^2 C_F f_\epsilon \int \frac{d^d k}{(2\pi)^d} \frac{1}{[-\bar{n} \cdot k]} \frac{\bar{n} \cdot (p_2 - k)}{[(p_2 - k)^2]} \frac{1}{k^2 - M^2}$$

Integrate over k^+ by contours

$$I_n = -2 \frac{C_F \alpha}{4\pi} \mu^{2\epsilon} e^{\epsilon \gamma_E} \Gamma(\epsilon) \int_0^1 dz \frac{1-z}{z} \left[M^2(1-z) - p_2^2 z(1-z) \right]^{-\epsilon}$$

For $z \to 0$ this integral diverges for $M \neq 0$, even if $p_2^2 \neq 0$.
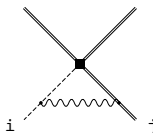If $M = 0$, $z^{-\epsilon}$ from the $p_2^2 \neq 0$ term regulates the integral.

Analytic Regulator

Need to regulate the SCET diagrams. Analytic regulator

$$\frac{1}{(p_i - k)^2 - m_i^2} \quad \rightarrow \quad \frac{(-\nu_i^2)^{\delta_i}}{\left[(p_i - k)^2 - m_i^2\right]^{1+\delta_i}} .$$

The above regulates the collinear fermion denominators.

If particle j interacts with a n_i collinear gluon, and becomes off-shell:

$$\frac{1}{(p_j + k)^2 - m_j^2} \rightarrow \frac{1}{2p_j \cdot k} \rightarrow \frac{p_j \cdot \epsilon}{p_j \cdot k} = \frac{n_j \cdot \epsilon}{n_j \cdot k}$$
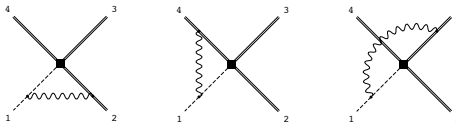
where k is n_i-collinear

Wilson Lines

$$
\begin{aligned}
k^{\mu} &= \frac{1}{2} \left(\bar{n}_i \cdot k \right) n_i^{\mu} \\
\epsilon^{\mu} &= \frac{1}{2} \left(\bar{n}_i \cdot \epsilon \right) n_i^{\mu} \\
p_j^{\mu} &= \frac{1}{2} \left(\bar{n}_j \cdot p_j \right) n_j^{\mu}
\end{aligned}
$$

$$
\frac{n_j \cdot \epsilon}{n_j \cdot k} = \frac{\bar{n}_i \cdot \epsilon}{\bar{n}_i \cdot k} \frac{n_i \cdot n_j}{n_i \cdot n_j} = \frac{\bar{n}_i \cdot \epsilon}{\bar{n}_i \cdot k}
$$

and only depends on i, and the color of Wilson line particle j.

Multiple particles:

Wilson lines transforming as T_2, T_3 and T_4 combine to form a Wilson line transforming as $T_{\bar{1}}$, *provided the operator is color singlet.* This gives

$$\left[W_1^\dagger \xi_1 \right]$$

W_1 contains T_1 and $\bar{n}_1$.

Bauer, Fleming, Pirjol, Rothstein, Stewart

$$\frac{n_j \cdot \epsilon}{n_j \cdot k} \to \frac{n_j \cdot \epsilon}{(n_j \cdot k)^{1+\delta}} = \frac{\bar{n}_i \cdot \epsilon}{(\bar{n}_i \cdot k)^{1+\delta}} \frac{n_i \cdot n_j}{(n_i \cdot n_j)^{1+\delta}} = \frac{\bar{n}_i \cdot \epsilon}{(\bar{n}_i \cdot k)^{1+\delta}} \frac{1}{(n_i \cdot n_j)^{\delta}}$$

The j dependence no longer cancels. Thus the identities which allowed one to combine all the n_i-collinear emissions into a single Wilson line in the $\bar{n}_i$ direction no longer hold.
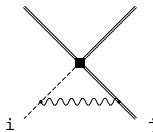
Good to have a nicer regulator.

Δ Regulator

$$\frac{1}{(p_i + k)^2 - m_i^2} \to \frac{1}{(p_i + k)^2 - m_i^2 - \Delta_i}$$

This regulator can be implemented at the level of the Lagrangian, since it corresponds to a shift in the particle mass. The on-shell condition remains $p_i^2 = m_i^2$.

If particle j interacts with a n_i collinear gluon, and becomes off-shell:

$$\frac{1}{(p_j + k)^2 - m_j^2 - \Delta_j} \to \frac{1}{2p_j \cdot k - \Delta_j}$$

where k is n_i-collinear

Δ Regulator

$$k^\mu = \frac{1}{2} (\bar{n}_i \cdot k) \, n_i^\mu$$
$$p_j^\mu = \frac{1}{2} (\bar{n}_j \cdot p_j) \, n_j^\mu$$

$$\frac{1}{(p_j + k)^2 - m_j^2 - \Delta_j} \to \frac{1}{\frac{1}{2}(\bar{n}_i \cdot k)(\bar{n}_j \cdot p_j)(n_i \cdot n_j) - \Delta_j}$$

where k is n-collinear, so

$$\frac{\epsilon \cdot n_j}{k \cdot n_j} \to \frac{\epsilon \cdot \bar{n}_i}{k \cdot \bar{n}_i - \delta_{j,n_i}}, \qquad \delta_{j,n_i} \equiv \frac{2\Delta_j}{(n_i \cdot n_j)(\bar{n}_j \cdot p_j)}$$

Still depends on j through the regulator δ_{j,n_i}.

For the Sudakov problem, collinear fermions regulated by $\Delta_{1,2}$ and Wilson lines by $\delta_{1,n}$ and $\delta_{2,\bar{n}}$

$$\delta_1 \equiv \delta_{1,n} = \frac{2\Delta_1}{(n \cdot \bar{n})(n \cdot p_1)} = \frac{\Delta_1}{p_1^+}$$

$$\delta_2 \equiv \delta_{2,\bar{n}} = \frac{2\Delta_2}{(n \cdot \bar{n})(\bar{n} \cdot p_2)} = \frac{\Delta_2}{p_2^-}.$$

δ_1 transforms as p^- and δ_2 as p^+.

$$I_n = \frac{\alpha C_F}{4\pi} \left[\left(\frac{2}{\epsilon} - 2L_M \right) \left(1 + \log \frac{\delta_1}{p_2^-} \right) - \frac{\pi^2}{3} + 2 \right]$$

Graph for particle 2 depends on regulator for particle 1 and on Q^2.

(Also collinear wavefunction graphs)

For the soft diagram, one finds

$$
\begin{aligned}
I_s &= -2ig^2 C_F f_\epsilon \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^2 - M^2} \frac{1}{-n \cdot k - \delta_2} \frac{1}{-\bar{n} \cdot k - \delta_1} \\
&= \frac{\alpha C_F}{4\pi} \left[-\frac{2}{\epsilon^2} + \frac{2}{\epsilon} \log \frac{\delta_1 \delta_2}{\mu^2} + \mathsf{L}_M^2 - 2\mathsf{L}_M \log \frac{\delta_1 \delta_2}{\mu^2} + \frac{\pi^2}{6} \right]
\end{aligned}
$$

$$
\delta_1 \delta_2 = \frac{\Delta_1 \Delta_2}{Q^2}
$$

Zero Bin Subtractions

Stewart, AM

Lee and Sterman, Collins, Sterman, Idilbi and Mehen

There is a double counting because the $p \to 0$ collinear integral overlaps the soft integral. The $p \to 0$ limit is

$$
I_{n\emptyset} = -2ig^2 C_F f_\epsilon \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{[-\bar{n} \cdot k - \delta_1]} \frac{1}{[-n \cdot k - \Delta_2/p_2^-]} \frac{1}{k^2 - M^2}
$$

which is the same as the soft integral with $\delta_2 \to \Delta_2/p_2^-$.

$$
I_n - I_{n\emptyset} = \frac{\alpha C_F}{4\pi} \left[\frac{2}{\epsilon^2} - \frac{2}{\epsilon} \log \frac{\Delta_2}{\mu^2} + \frac{2}{\epsilon} - 2 \left(1 - \log \frac{\Delta_2}{\mu^2} \right) \mathsf{L}_M \right.
$$
$$
\left. - \mathsf{L}_M^2 - \frac{\pi^2}{2} + 2 \right]
$$

Depends only on M and Δ_2 — no longer depends on particle 1.

$$(I_n - I_{n\varnothing}) + (I_{\bar{n}} - I_{\bar{n}\varnothing}) + I_s + \text{wavefunction}$$

$$= \frac{\alpha C_F}{4\pi} \left[\frac{2}{\epsilon^2} + \frac{2}{\epsilon} \log \frac{\delta_1 \delta_2 \mu^2}{\Delta_1 \Delta_2} + \frac{3}{\epsilon} - L_M^2 - 2L_M \log \frac{\delta_1 \delta_2 \mu^2}{\Delta_1 \Delta_2} - 3L_M \right.$$
$$\left. - \frac{5\pi^2}{6} + \frac{9}{2} \right]$$

This simplifies to (using $\delta_1 = \Delta_1/p_1^+$, $\delta_2 = \Delta_2/p_2^-$)

$$\frac{\alpha C_F}{4\pi} \left[\frac{2}{\epsilon^2} - \frac{2}{\epsilon} L_Q + \frac{3}{\epsilon} - L_M^2 + 2L_Q L_M - 3L_M - \frac{5\pi^2}{6} + \frac{9}{2} \right]$$

This is the correct effective theory result, and when combined with the matching computation at Q correctly reproduces the known full-theory computation of the form-factor.

Note that without zero-bin subtractions, the effective theory result would be

$$I_n + I_{\bar{n}} + I_s + \text{wavefunction}$$

$$= \frac{\alpha C_F}{4\pi} \left[-\frac{2}{\epsilon^2} + \frac{2}{\epsilon} \log \frac{\Delta_1^2 \Delta_2^2}{Q^6 \mu^2} + \frac{3}{\epsilon} + \mathsf{L}_M^2 - 2\mathsf{L}_M \log \frac{\Delta_1^2 \Delta_2^2}{Q^6 \mu^2} - 3\mathsf{L}_M \right.$$
$$\left. -\frac{\pi^2}{2} + \frac{9}{2} \right]$$

- Wrong $1/\epsilon$ part — wrong anomalous dimension
- Wrong finite pieces
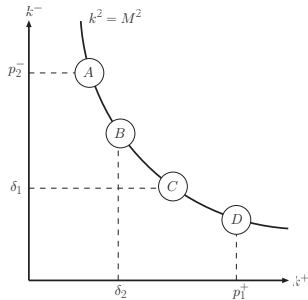- Regulator does not cancel
- Idilbi and Mehen pointed this out for QCD

The zero-bin subtraction $I_{n\varnothing} = I_s$ so

$$(I_n - I_{n\varnothing}) + (I_{\bar{n}} - I_{\bar{n}\varnothing}) + I_s = I_n + I_{\bar{n}} - I_s$$

Following Idilbi and Mehen, one can write down the combination

$$I_n + I_{\bar{n}} - I_s$$

- Evaluating it gives the correct EFT result
- Do not need any regulator (drop the Δ regulator)
- Method of regions — always add up regions. Here you have to subtract a region.

Δ regulator introduces new regulator dependent regions

- I_n from A and C $I_{\bar{n}}$ from D and B
- I_s from B and C
- $I_{n\emptyset}$ from B and C $I_n - I_{n\emptyset}$ from A and B
- $I_n + I_{\bar{n}} - I_s$ from regions A and B

Gauge Dependence

Also need zero-bin subtractions for gauge invariance

$$i\Delta_{\alpha\beta}(k) = \frac{1}{k^2 - M^2}\left[g_{\alpha\beta} + (\xi - 1)\frac{k_\alpha k_\beta}{k^2 - \xi M^2}\right]$$

In the full theory, the new ξ dependent contribution to the vertex graph, $I^{(\xi)}$, stemming from the second part of the gauge boson propagator is
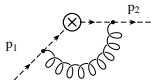
$$
\begin{aligned}
I^{(\xi)} &= \Gamma J_\xi \\
J_\xi &= -ig^2 C_F(\xi - 1)f_\epsilon \int \frac{d^d k}{(2\pi)^d}\frac{1}{k^2 - M^2}\frac{1}{k^2 - \xi M^2} \\
&= \frac{\alpha_s}{4\pi}C_F\left[\frac{\xi - 1}{\epsilon_{\mathrm{UV}}} - (\xi - 1)\log\frac{M^2}{\mu^2} + (\xi - 1) - \xi\log\xi\right]
\end{aligned}
$$

- High Scale matching is gauge invariant:

$$V \to V + J_\xi, \qquad W \to W + J_\xi, \qquad V - W \to V - W$$

- EFT without zero-bin subtractions is gauge dependent:

$$I_n \to I_n + J_\xi \qquad I_{\bar{n}} \to I_{\bar{n}} + J_\xi \qquad I_s \to I_s + J_\xi$$

$$W_n \to W_n + J_\xi \qquad W_{\bar{n}} \to W_{\bar{n}} + J_\xi \qquad W_s \to W_s + J_\xi$$

$$\delta \left(I_n - \tfrac{1}{2} W_n \right) = \tfrac{1}{2} J_\xi \quad \delta \left(I_{\bar{n}} - \tfrac{1}{2} W_{\bar{n}} \right) = \tfrac{1}{2} J_\xi \quad \delta \left(I_s - W_s \right) = 0$$

- Fixed by the zero-bin subtractions:

$$\delta \left(I_{n\varnothing} - \tfrac{1}{2} W_{n\varnothing} \right) = \tfrac{1}{2} J_\xi \quad \delta \left(I_{\bar{n}\varnothing} - \tfrac{1}{2} W_{\bar{n}\varnothing} \right) = \tfrac{1}{2} J_\xi$$

Factorization

$I_n - I_{n\emptyset}$ does not need any regulator for the Wilson lines, only for the collinear fermion.

$$\frac{\epsilon \cdot n_j}{k \cdot n_j} \;\to\; \frac{\epsilon \cdot \bar{n}_i}{k \cdot \bar{n}_i - \delta_{j,n_i}} \to \frac{\epsilon \cdot n_j}{k \cdot n_j}$$
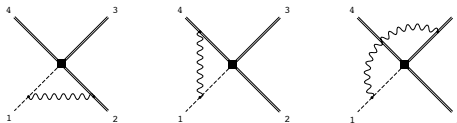
Wilson lines transforming as T_2, T_3 and T_4 combine to form a Wilson line transforming as $T_{\bar{1}}$, *provided the operator is color singlet.*

$$\left[W_1^\dagger \xi_1 \right]_{\emptyset}$$

W_1 contains T_1 and $\bar{n}_1$ and does not depend on the other particles, provided collinear graphs are zero-bin subtracted.

Factorization

Important result of Aybat, Dixon, Sterman; Dixon, Magnea, Sterman and Dixon

$$\text{Soft} \propto \sum_{\langle ij \rangle} \mathbf{T}_i \cdot \mathbf{T}_j$$

even at two-loop order.

Factorization properties combined with the color structure imply that one can use the sum on pairs formula to two-loop order. Chiu et al.

Thus as discussed at SCET08, the Sudakov computation allows one to compute an arbitrary *n*-particle process by summing the Sudakov result over pairs with the $\mathbf{T}_i \cdot \mathbf{T}_j$ group theory factor.

Can now compute *all* standard model processes at high energy — missing piece was electroweak gauge bosons (Talk by Kelley).

Conclusions

- Need zero-bin subtractions — not zero for massive gauge bosons
- This is not a regulator issue — $I_n + I_{\bar{n}} - I_s$ does not need an IR regulator.
- Required for collinear gauge invariance
- Restores factorization between collinear and soft
- Can compute radiative corrections to any hard scattering process in the standard model.