IR Divergences of Gauge-Theory Amplitudes and Resummation for LHC

Matthias Neubert

Johannes Gutenberg University Mainz

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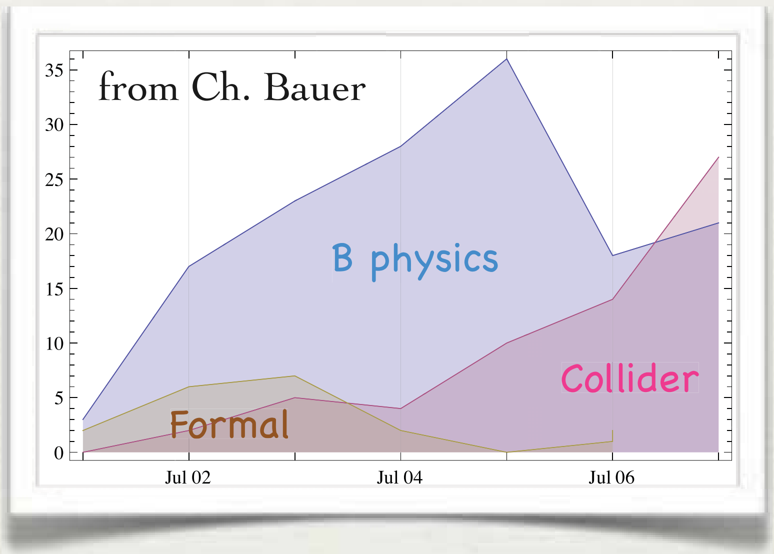
Thomas Becher & MN: arXiv:0901.0722 and arXiv:0903.1126

SCET applications to collider physics

- In recent years, increasing focus on applying SCET to collider physics:
  - really large energies, small power corrections
  - relevance to LHC



- Can we make an impact ?
  - early work was rederiving known results
  - only recently, unsolved problems are being attacked

Examples of recent applications

- Drell-Yan rapidity distribution — Becher, MN, Xu 2007
- $e^+e^- \to e^+e^-$ at NNLO for $m_e^2 < s, |t|$ — Becher, Melnikov 2007
- Factorization for jet production — Bauer, Hornig, Tackmann 2008
- Angularities in e^+e^- — Hornig, Lee, Ovanesyan 2009
- Explanation of large K-factor for inclusive Higgs production (talk by T. Becher for L. Yang) — Ahrens, Becher, MN, Yang 2008
- Resummation for heavy color-octet production — Idilbi, Kim 2009
- Precision top mass determination from event shapes in e^+e^- — Fleming, Hoang, Mantry, Stewart 2008
- Electroweak Sudakov resummation — Chiu, Kelley, Manohar 2007
- W-pair production in e^+e^- near threshold — Beneke, Falgari, Schwinn, Signer, Zanderighi 2007; Actis, Beneke, Falgari, Schwinn 2008

Towards n-jet processes:
IR divergencies of scattering amplitudes

IR singularities

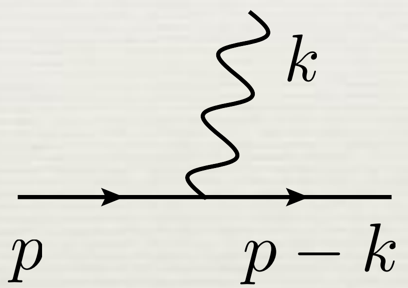
- On-shell parton scattering amplitudes in gauge theories contain IR divergences from soft and collinear loop momenta
- IR singularities cancel between real and virtual contributions

Bloch, Nordsieck 1937
Kinoshita 1962; Lee, Nauenberg 1964

- Nevertheless interesting:
  - resummation of large Sudakov logarithms remaining after cancellation of divergences (very relevant for LHC physics!)
  - check on multi-loop calculations

IR singularities in QED

- Singularities arise from soft photon emission (for $m_e \neq 0$); eikonal approximation:



$$\ldots \frac{\not p - \not k + m}{(p-k)^2 - m^2}\, \gamma_\mu\, u(p)$$

$$\approx\ \ldots\ u(p)\, \boxed{\frac{p_\mu}{p \cdot k}}$$

- IR divergent part is a multiplicative factor
- Higher-order terms obtained by exponentiating leading-order soft contribution

Yennie, Frautschi, Suura 1961
Weinberg 1965

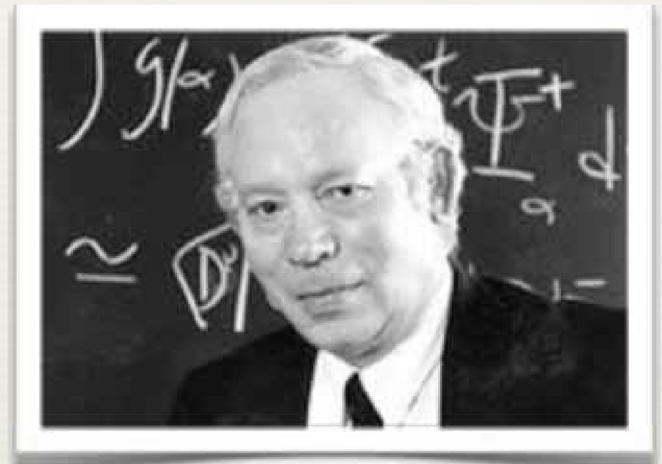
IR singularities in QCD

- Much more complicated
  - soft and collinear singularities
  - gluons carry color charge, hence soft emissions do not simply exponentiate
  - but only a restricted set of higher-order contributions can appear (non-abelian exponentiation theorem) Gatheral 1983; Frenkel, Taylor 1984
- Form long time, explicit form of IR poles was only understood at two-loop order Catani 1998

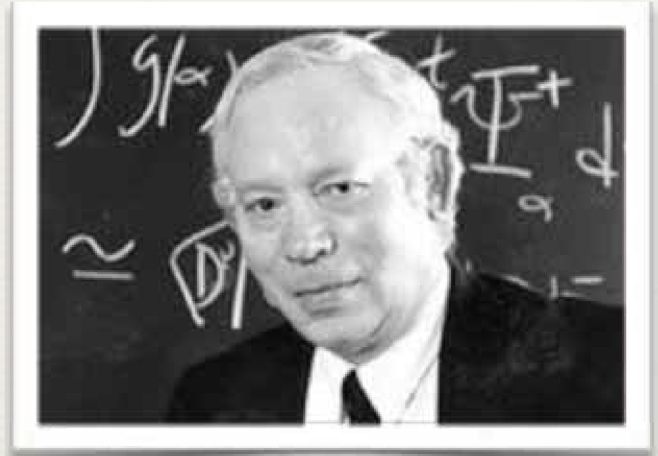
IR singularities in QCD

Difficulty of the problem eloquently formulated in pioneering work on QED by Weinberg:

S. Weinberg, Phys. Rev. 140B, 516 (1965)



IR singularities in QCD



Difficulty of the problem eloquently formulated in pioneering work on QED by Weinberg:

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"… But these remarks do not apply to theories involving charged massless particles. In such theories (including the Yang-Mills theory) a soft photon emitted from an external line can itself emit a pair of soft charged massless particles, which themselves emit soft photons, and so on, building up a cascade of soft massless particles each of which contributes an infra-red divergence. **The elimination of such complicated interlocking infra-red divergences would certainly be a Herculean task, and might not even be possible.**

… Perhaps it would not be too much to suggest that it is the infra-red divergences that prohibit the existence of Yang-Mills quanta, or other charged massless particles."

Color-space formalism

- Represent amplitudes as vectors in color space:

Catani, Seymour 1996

$$|c_1, c_2, \ldots, c_n\rangle$$

color index of first parton

- Color generator for i^th parton $\boldsymbol{T}_i^a \, |c_1, c_2, \ldots, c_n\rangle$ acts like a matrix:
  - t^a matrix for quarks, f^{abc} for gluons
  - product $\boldsymbol{T}_i \cdot \boldsymbol{T}_j = \sum_a \boldsymbol{T}_i^a \boldsymbol{T}_j^a$ (commutative)
  - charge conservation $\sum_i \boldsymbol{T}_i^a = 0$ implies:

$$\sum_{(i,j)} \boldsymbol{T}_i \cdot \boldsymbol{T}_j = -\sum_i \boldsymbol{T}_i^2 = -\sum_i C_i$$

i≠j

C_F or C_A

Catani's two-loop formula (1998)

("... beautiful, yet mysterious ...")

- Specifies IR singularities of dimensionally regularized n-parton amplitudes at two loops:

$$\left[1 - \frac{\alpha_s}{2\pi}\,\boldsymbol{I}^{(1)}(\epsilon) - \left(\frac{\alpha_s}{2\pi}\right)^2 \boldsymbol{I}^{(2)}(\epsilon) + \ldots\right] |\mathcal{M}_n(\epsilon, \{\underline{p}\})\rangle = \text{finite}$$

amplitude is vector in color space

with

$$\boldsymbol{I}^{(1)}(\epsilon) = \frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} \sum_i \left(\frac{1}{\epsilon^2} + \frac{g_i}{\boldsymbol{T}_i^2}\,\frac{1}{\epsilon}\right) \sum_{j\neq i} \frac{\boldsymbol{T}_i\cdot\boldsymbol{T}_j}{2} \left(\frac{\mu^2}{-s_{ij}}\right)^\epsilon$$

$(p_i + p_j)^2$

$$\boldsymbol{I}^{(2)}(\epsilon) = \frac{e^{-\epsilon\gamma_E}\,\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)} \left(K + \frac{\beta_0}{2\epsilon}\right) \boldsymbol{I}^{(1)}(2\epsilon) - \frac{1}{2}\,\boldsymbol{I}^{(1)}(\epsilon)\left(\boldsymbol{I}^{(1)}(\epsilon) + \frac{\beta_0}{\epsilon}\right) + \boldsymbol{H}^{(2)}_{\text{R.S.}}(\epsilon)$$

unspecified

- Later derivation using factorization properties and IR evolution equation for form factor

Sterman, Tejeda-Yeomans 2003

All-order generalization

- Have argued that IR divergences in d=4-2ε can be absorbed into a multiplicative factor **Z** (a matrix in color space), which derives from an anomalous-dimension matrix:

Becher, MN 2009

$$|\mathcal{M}_n(\{\underline{p}\},\mu)\rangle = \lim_{\epsilon\to 0} \boldsymbol{Z}^{-1}(\epsilon,\{\underline{p}\},\mu)\,|\mathcal{M}_n(\epsilon,\{\underline{p}\})\rangle$$

finite amplitude!

$$\boldsymbol{Z}(\epsilon,\{\underline{p}\},\mu) = \mathbf{P}\exp\left[\int_\mu^\infty \frac{\mathrm{d}\mu'}{\mu'}\,\boldsymbol{\Gamma}(\{\underline{p}\},\mu')\right]$$

- Corresponding RG evolution equation:

$$\frac{d}{d\ln\mu}\,|\mathcal{M}_n(\{\underline{p}\},\mu)\rangle = \boldsymbol{\Gamma}(\{\underline{p}\},\mu)\,|\mathcal{M}_n(\{\underline{p}\},\mu)\rangle$$

⇒ can be used to resum Sudakov logarithms

All-order generalization

- Anomalous dimension is conjectured to be extremely simple:

color charges

anom. dimensions, known to three-loop order

$$\boldsymbol{\Gamma}(\{\underline{p}\},\mu) = \sum_{(i,j)} \frac{\boldsymbol{T}_i \cdot \boldsymbol{T}_j}{2}\, \gamma_{\rm cusp}(\alpha_s)\, \ln\frac{\mu^2}{-s_{ij}} + \sum_i \gamma^i(\alpha_s)$$

sum over pairs i≠j of partons

$(p_i + p_j)^2$

  - simple structure, reminiscent of QED
  - IR poles determined by color charges and momenta of external partons (semi-classical)
  - color dipole correlations, like at one-loop order

All-order generalization

- Result is surprising, as it implies amazing cancellations to occur in multi-loop calculations
- Normally, expect that complexity of L-loop anomalous dimension equals that of (L-1)-loop finite terms, which are known to contain complicated color and momentum structures!
- Here different: pole terms are protected by soft-collinear factorization theorem!

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Z factor to three loops

- Explicit result:

d-dimensional β-function

$$\ln \boldsymbol{Z}(\epsilon, \{\underline{p}\}, \mu) = \int_0^{\alpha_s} \frac{d\alpha}{\alpha} \frac{1}{2\epsilon - \beta(\alpha)/\alpha} \left[\boldsymbol{\Gamma}(\{\underline{p}\}, \mu, \alpha) + \int_0^{\alpha} \frac{d\alpha'}{\alpha'} \frac{\Gamma'(\alpha')}{2\epsilon - \beta(\alpha')/\alpha'} \right]$$

where

$$\Gamma'(\alpha_s) \equiv \frac{\partial}{\partial \ln \mu} \boldsymbol{\Gamma}(\{\underline{p}\}, \mu, \alpha_s) = -\gamma_{\text{cusp}}(\alpha_s) \sum_i C_i$$

- Perturbative expansion:

all coefficients known!

$$\begin{aligned} \ln \boldsymbol{Z} &= \frac{\alpha_s}{4\pi} \left(\frac{\Gamma'_0}{4\epsilon^2} + \frac{\boldsymbol{\Gamma}_0}{2\epsilon} \right) + \left(\frac{\alpha_s}{4\pi} \right)^2 \left[-\frac{3\beta_0 \Gamma'_0}{16\epsilon^3} + \frac{\Gamma'_1 - 4\beta_0 \boldsymbol{\Gamma}_0}{16\epsilon^2} + \frac{\boldsymbol{\Gamma}_1}{4\epsilon} \right] \\ &+ \left(\frac{\alpha_s}{4\pi} \right)^3 \left[\frac{11\beta_0^2 \, \Gamma'_0}{72\epsilon^4} - \frac{5\beta_0 \Gamma'_1 + 8\beta_1 \Gamma'_0 - 12\beta_0^2 \, \boldsymbol{\Gamma}_0}{72\epsilon^3} + \frac{\Gamma'_2 - 6\beta_0 \boldsymbol{\Gamma}_1 - 6\beta_1 \boldsymbol{\Gamma}_0}{36\epsilon^2} + \frac{\boldsymbol{\Gamma}_2}{6\epsilon} \right] + \dots \end{aligned}$$

⇒ exponentiation yields **Z** factor at three loops!

Checks

- Expression for IR pole terms agrees with all known perturbative results:
  - 3-loop quark and gluon form factors, which determine the functions $\gamma^{q,g}(\alpha_s)$
    Moch, Vermaseren, Vogt 2005
  - 2-loop 3-jet qqg amplitude
    Garland, Gehrmann et al. 2002
  - 2-loop 4-jet amplitudes
    Anastasiou, Glover et al. 2001
    Bern, De Freitas, Dixon 2002, 2003
  - 3-loop 4-jet amplitudes in N=4 super Yang-Mills theory in planar limit
    Bern et al. 2005, 2007

Catani's result

- Comparison with Catani's formula at two loops yields explicit expression for 1/ε pole term:

$$\boldsymbol{H}^{(2)}_{\text{R.S.}}(\epsilon) = \frac{1}{16\epsilon} \sum_i \left(\gamma_1^i - \frac{1}{4}\, \gamma_1^{\text{cusp}}\, \gamma_0^i + \frac{\pi^2}{16}\, \beta_0\, C_i \right)$$

$$+ \frac{i f^{abc}}{24\epsilon} \sum_{(i,j,k)} \boldsymbol{T}_i^a\, \boldsymbol{T}_j^b\, \boldsymbol{T}_k^c \ln\frac{-s_{ij}}{-s_{jk}} \ln\frac{-s_{jk}}{-s_{ki}} \ln\frac{-s_{ki}}{-s_{ij}}$$

- Non-trivial color structure only arises since his operators are not defined in a minimal scheme

see also: Mert Aybat, Dixon, Sterman 2006

- Our result confirms an earlier conjecture for the form of this term Bern, Dixon, Kosower 2004

Key ideas and arguments supporting our conjecture

Misconception

- Conventional thinking is that UV and IR divergences are of totally different nature:
  - UV divergences absorbed into renormalization of parameters of theory; structure constrained by RG equations
  - IR divergences arise in unphysical calculations; cancel between virtual corrections and real emissions
- In fact, IR divergences can be mapped onto UV divergences of operators in effective field theory!

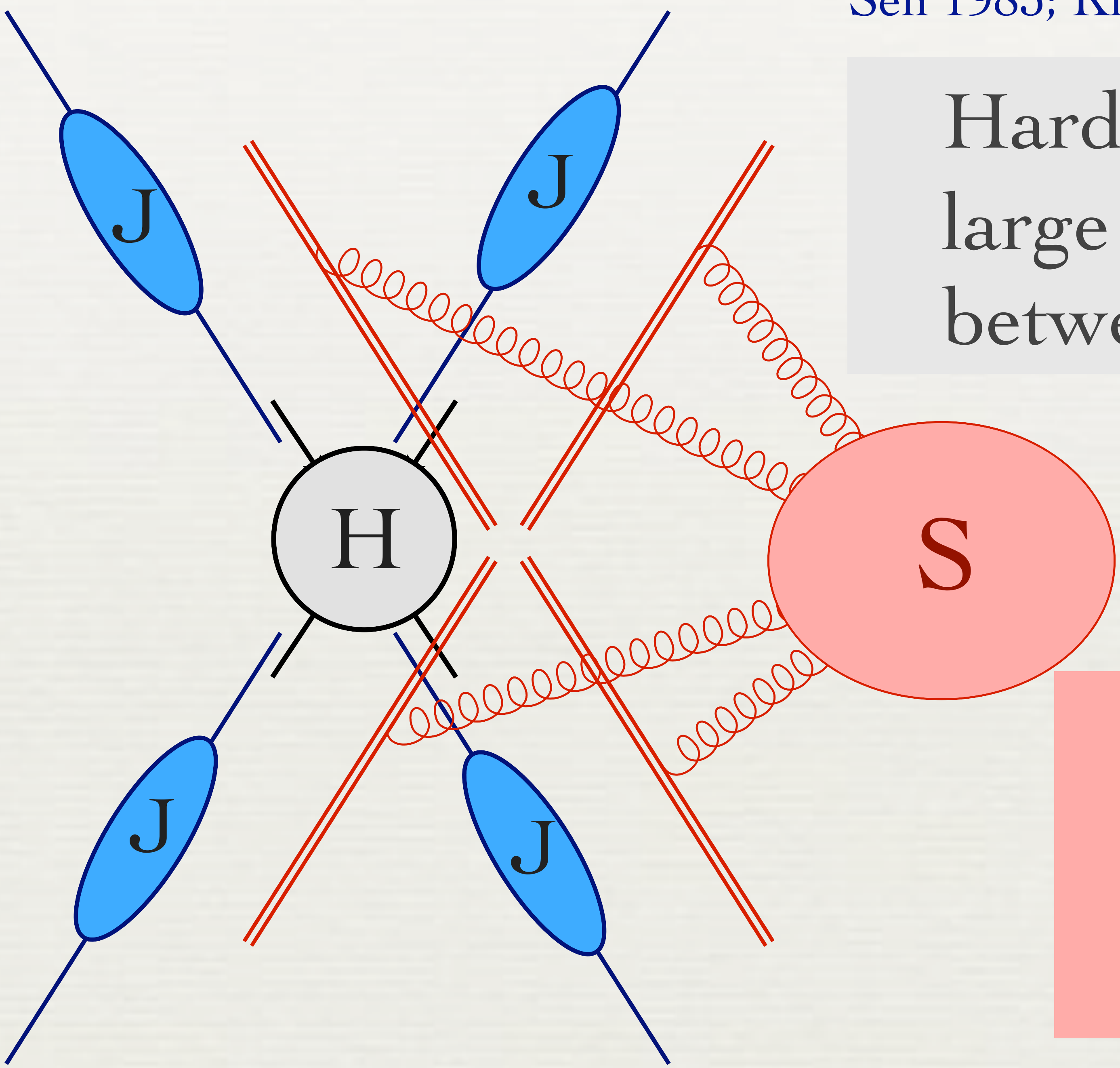
Interpretation and derivation

- In our case, $\boldsymbol{\Gamma}$ is the anomalous-dimension matrix of n-jet operators in SCET, and $\boldsymbol{Z}$ is the associated matrix of renormalization factors
- Will now discuss structure of SCET for n-jet processes and constraints on anomalous dimension $\boldsymbol{\Gamma}$ arising from
  - charge conservation $\sum_i \boldsymbol{T}_i = 0$
  - soft-collinear factorization
  - non-abelian exponentiation
  - consistency with collinear limits

Becher, MN, arXiv:0903.1126

Soft-collinear factorization

Sen 1983; Kidonakis, Oderda, Sterman 1998



Hard function H depends on large momentum transfers s_{ij} between jets

Soft function S depends on scales $\Lambda_{ij}^2 = \dfrac{M_i^2 M_j^2}{s_{ij}}$

Jet functions $J_i = J_i\,(M_i^2)$

SCET for n-jet processes

- n different types of collinear quark and gluon fields (→ jet functions J_i), interacting only via soft fields (soft function S)
  - operator definitions for J_i and S
- Hard contributions ($Q \sim \sqrt{s}$) are integrated out and absorbed into Wilson coefficients:

$$\mathcal{H}_n = \sum_i \mathcal{C}_{n,i}(\mu)\, O_{n,i}^{\text{ren}}(\mu)$$

Bauer, Schwartz 2006

- Scale dependence controlled by RGE:

$$\frac{d}{d\ln\mu}\,|\mathcal{C}_n(\{\underline{p}\},\mu)\rangle = \mathbf{\Gamma}(\mu,\{\underline{p}\})\,|\mathcal{C}_n(\{\underline{p}\},\mu)\rangle$$

anomalous-dimension matrix

On-shell parton scattering amplitudes

- Hard functions C_n can be obtained by setting the jet masses to zero: jet and soft functions become scaleless, loop corrections vanish.
- One obtains:

renormalization factor
(minimal subtraction of IR poles)

$$|\mathcal{C}_n(\{\underline{p}\},\mu)\rangle = \lim_{\epsilon\to 0} \boldsymbol{Z}^{-1}(\epsilon,\{\underline{p}\},\mu)\,|\mathcal{M}_n(\epsilon,\{\underline{p}\})\rangle$$

Becher, MN 2009

where

$$\boldsymbol{\Gamma} = -\frac{d\ln\boldsymbol{Z}}{d\ln\mu}$$

- IR poles of scattering amplitudes mapped onto UV poles of n-jet SCET operators
- Multiplicative subtraction, controlled by RG

Constraints from soft-collinear factorization

Factorization constraint on Γ

- Operator matrix elements must evolve in the same way as hard matching coefficients, such that physical observables are scale independent
- SCET decoupling transformation then implies (with $\Lambda_{ij}^2 = \dfrac{M_i^2 M_j^2}{s_{ij}}$):

trivial color structure

$$\boldsymbol{\Gamma}(s_{ij}) = \boldsymbol{\Gamma}_s(\Lambda_{ij}^2) + \sum_i \Gamma_c^i(M_i^2)\,\mathbf{1}$$

M_i dependence must cancel!

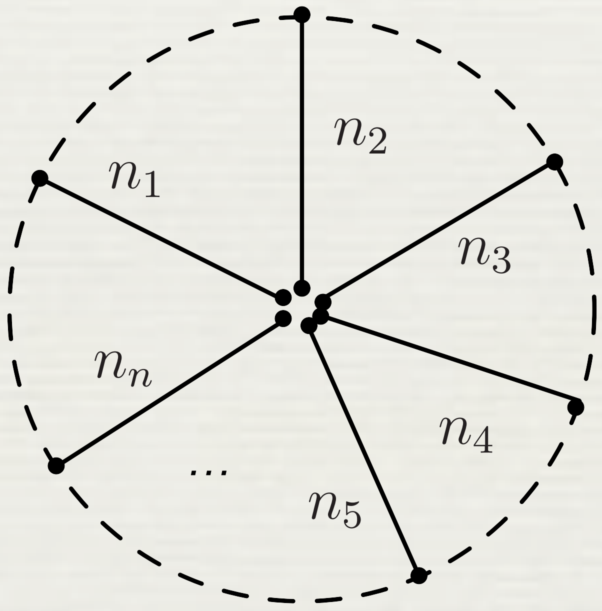
  - suggests logarithmic dependence on s_{ij} and M_i^2
  - $\boldsymbol{\Gamma}$ and $\boldsymbol{\Gamma}_S$ must have same color structure

Soft function

- SCET decoupling transformation removes soft interactions among collinear fields and absorbs them into soft Wilson lines

$n_i \sim p_i$ light-like reference vector

$$\boldsymbol{S}_i = \mathbf{P} \exp \left[ig \int_{-\infty}^{0} dt\, n_i \cdot A_a(tn_i)\, T_i^a \right]$$



- For n-jet operator one gets:

$$\mathcal{S}(\{\underline{n}\}, \mu) = \langle 0 | \boldsymbol{S}_1(0) \dots \boldsymbol{S}_n(0) | 0 \rangle = \exp(\tilde{\mathcal{S}}(\{\underline{n}\}, \mu))$$

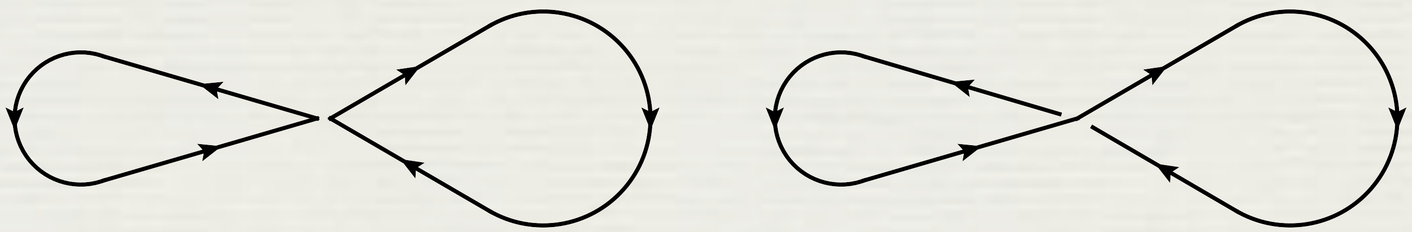
"Mercedes star operators"

Renormalization of Wilson loops

- Wilson loops containing singular points (cusps or cross points) require UV subtractions

Polyakov 1980; Brandt, Neri, Sato 1981

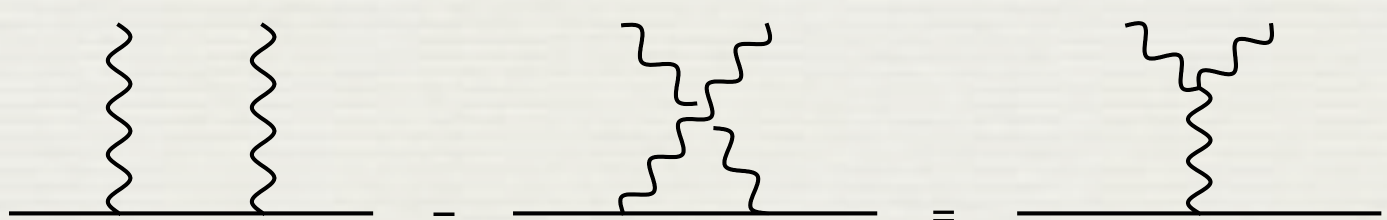
- For single cusp formed by tangent vectors n_1 and n_2, renormalization factor depends on cusp angle β_{12} defined as $$\cosh\beta_{12} = \frac{n_1 \cdot n_2}{\sqrt{n_1^2\, n_2^2}}$$
- More generally, sets of related Wilson loops mix under renormalization, with $\mathbf{Z}_s$ matrix depending on all relevant cusp angles



Non-abelian exponentiation

Gatheral 1983; Frenkel and Taylor 1984

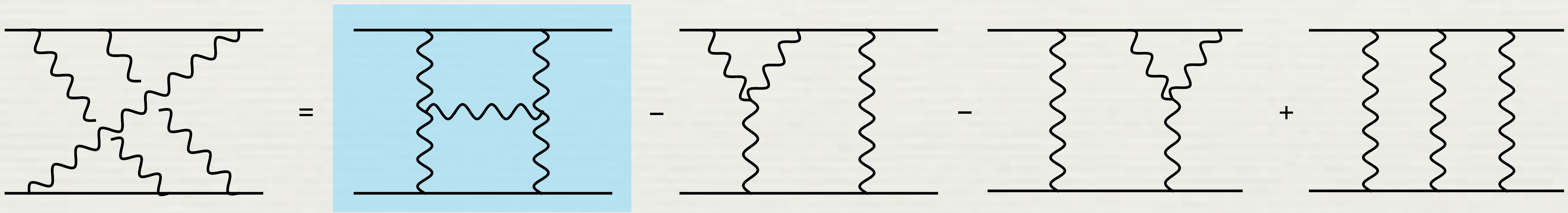
- Purely virtual amplitudes in eikonal (i.e., soft-gluon) approximation can be written as exponentials of simpler quantities, which receive contributions only from Feynman diagrams whose color weights are "color-connected" (or "maximally non-abelian")
- Color-weight graphs associated with each Feynman diagram can be simplified using the Lie commutator relation:



$$\boldsymbol{T}^a\boldsymbol{T}^b - \boldsymbol{T}^b\boldsymbol{T}^a = if^{abc}\boldsymbol{T}^c$$

Non-abelian exponentiation

- Use this to decompose any color-weight graph into a sum over products of connected webs, defined as a connected set of gluon lines (not counting crossed lines as being connected)

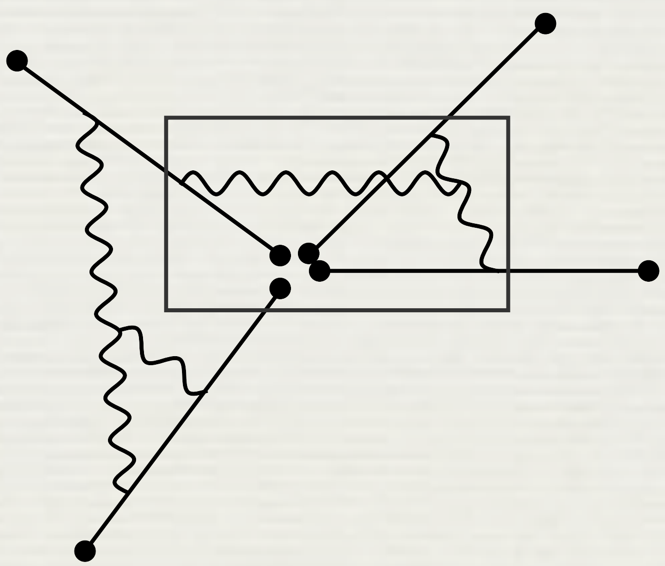


single connected web
"maximally nonabelian"

- Only color structures consisting of a single connected web contribute to the exponent $\tilde{S}$

Non-abelian exponentiation

- Single connected webs are two-particle irreducible with respect to Wilson lines
- In our case the gluons of the web can connect to more than two Wilson lines
- Fact that only single connected webs contribute to $\ln \mathbf{Z}_s$ and $\mathbf{\Gamma}_s$, while products of webs contribute to $\mathbf{Z}_s$, is in analogy with structure of nested UV divergences in QFT (Zimmermann's forest formula)



Light-like Wilson lines

- For large values of cusp angle β_{12}, anomalous dimension associated with a cusp or cross point grows linearly with β_{12}, which is then approximately equal to $\ln(2n_1 \cdot n_2/\sqrt{n_1^2\, n_2^2})$

Korchemsky, Radyushkin 1987

- Cusp angle diverges when one or both segments approach the light-cone:

$$\Gamma(\beta_{12}) \stackrel{n_{1,2}^2 \to 0}{\longrightarrow} \Gamma^i_{\rm cusp}(\alpha_s)\, \ln \frac{\mu^2}{\Lambda_s^2} + \dots$$

Korchemskaya, Korchemsky 1992

- Presence of single logarithm characteristic for Sudakov problems (double logs)

Light-like Wilson lines

- In SCET, this feature has been found for 2-jet operators of quarks and gluons:

Manohar 2003
Becher, MN 2006
Ahrens, Becher, MN, Yang 2008

$$\Gamma_{2-\text{jet}} = -\Gamma^i_{\text{cusp}}(\alpha_s)\,\ln\frac{\mu^2}{-s} + 2\gamma^i(\alpha_s)$$

- Appearance of logarithms of hard scale is perplexing, but can be understood based on scale correlation $\mu_c^2 \sim \mu_h\,\mu_s$, which implies:

$$\ln\frac{\mu^2}{\mu_h^2} = 2\ln\frac{\mu^2}{\mu_c^2} - \ln\frac{\mu^2}{\mu_s^2}$$

- For such a rewriting to be possible, the anomalous dimension must depend single-logarithmically on momenta

Light-like Wilson lines

- Introducing IR regulators $p_i^2 \neq 0$ to define the soft and collinear scales, we obtain:

$$\beta_{ij} = L_i + L_j - \ln \frac{\mu^2}{-s_{ij}}$$

$$\beta_{ij} = \ln \frac{-s_{ij}\,\mu^2}{(-p_i^2)(-p_j^2)}$$

soft log

hard log

$$L_i = \ln \frac{\mu^2}{-p_i^2}$$

collinear log

Soft anomalous-dimension matrix

- Decompositions:

$$\mathbf{\Gamma}(\{\underline{p}\},\mu) = \mathbf{\Gamma}_s(\{\underline{\beta}\},\mu) + \sum_i \Gamma_c^i(L_i,\mu)$$

$$\Gamma_c^i(L_i) = -\Gamma_{\text{cusp}}^i(\alpha_s)\, L_i + \gamma_c^i(\alpha_s)$$

- Key equation:

see also: Gardi, Magnea, arXiv:0901.1091

$$\boxed{\frac{\partial \mathbf{\Gamma}_s(\{\underline{s}\},\{\underline{L}\},\mu)}{\partial L_i} = \Gamma_{\text{cusp}}^i(\alpha_s)}$$

- Enforces linearity in cusp angles β_{ij} and significantly restricts color structures

Soft anomalous-dimension matrix

- Only exception would be a more complicated dependence on conformal cross ratios, which are independent of collinear scales:

$$\beta_{ijkl} = \beta_{ij} + \beta_{kl} - \beta_{ik} - \beta_{jl} = \ln \frac{(-s_{ij})(-s_{kl})}{(-s_{ik})(-s_{jl})}$$

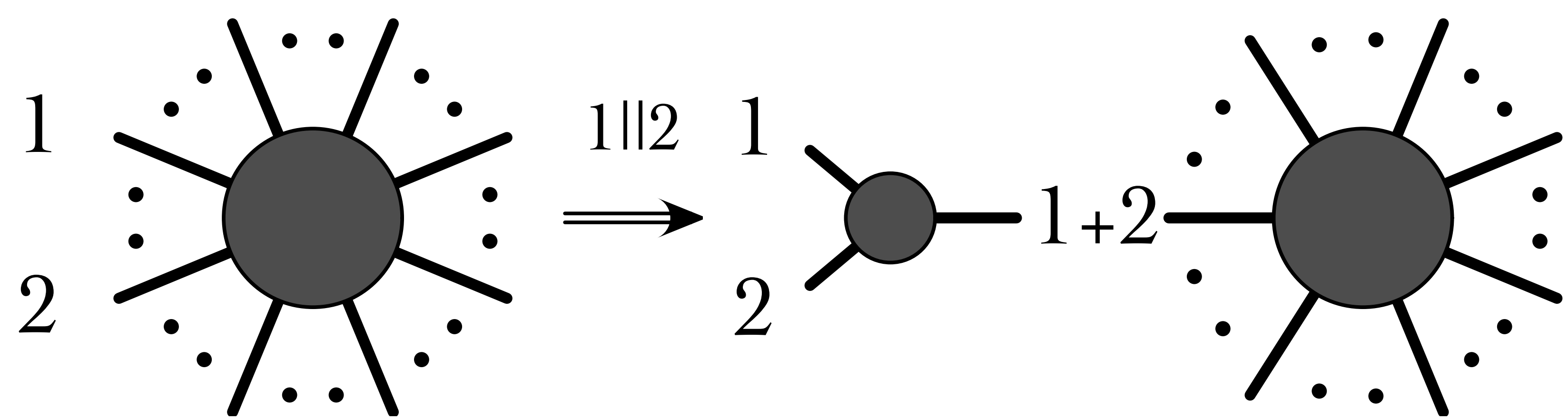
Gardi, Magnea 2009

- Can be excluded using other arguments, such as consistency with collinear limits

Consistency with collinear limits

- When two partons become collinear, an n-point amplitude M_n reduces to an (n-1)-parton amplitude times a splitting function: Berends, Giele 1989; Mangano, Parke 1991 Kosower 1999; Catani, de Florian, Rodrigo 2003

$$|\mathcal{M}_n(\{p_1, p_2, p_3, \dots, p_n\})\rangle = \mathbf{Sp}(\{p_1, p_2\})\,|\mathcal{M}_{n-1}(\{P, p_3, \dots, p_n\})\rangle + \dots$$



$$\mathbf{\Gamma}_{\mathrm{Sp}}(\{p_1, p_2\}, \mu) = \mathbf{\Gamma}(\{p_1, \dots, p_n\}, \mu) - \mathbf{\Gamma}(\{P, p_3 \dots, p_n\}, \mu)\Big|_{\boldsymbol{T}_P \to \boldsymbol{T}_1 + \boldsymbol{T}_2}$$

- $\mathbf{\Gamma}_{\mathrm{Sp}}$ must be independent of momenta and colors of partons 3, ..., n

Becher, MN 2009

Consistency check

- The form we propose is consistent with factorization in the collinear limit:

$$\mathbf{\Gamma}_{\rm Sp}(\{p_1,p_2\},\mu) = \mathbf{\Gamma}(\{p_1,\dots,p_n\},\mu) - \mathbf{\Gamma}(\{P,p_3\dots,p_n\},\mu)\big|_{\boldsymbol{T}_P\to\boldsymbol{T}_1+\boldsymbol{T}_2}$$

$$\mathbf{\Gamma}_{\rm Sp}(\{p_1,p_2\},\mu) = \gamma_{\rm cusp}\left[\boldsymbol{T}_1\cdot\boldsymbol{T}_2\,\ln\frac{\mu^2}{-s_{12}} + \boldsymbol{T}_1\cdot(\boldsymbol{T}_1+\boldsymbol{T}_2)\,\ln \underset{\substack{\uparrow\\ \text{momentum fraction of parton 1}}}{z} + \boldsymbol{T}_2\cdot(\boldsymbol{T}_1+\boldsymbol{T}_2)\,\ln(1-z)\right] + \gamma^1+\gamma^2-\gamma^P\,,$$

- But this would not work if $\mathbf{\Gamma}$ would involve terms of higher powers in color generators $\boldsymbol{T}_i$ or momentum variables
- A very strong constraint (new)!

$$\boldsymbol{\Gamma}_s(\{\underline{\beta}\},\mu) \stackrel{?}{=} -\sum_{(i,j)} \frac{\boldsymbol{T}_i\cdot\boldsymbol{T}_j}{2}\,\gamma_{\rm cusp}(\alpha_s)\,\beta_{ij} + \sum_i \gamma_s^i(\alpha_s)$$

Diagrammatic analysis of the soft anomalous-dimension matrix

Existing results

- Our conjecture implies for the soft anomalous-dimension matrix:

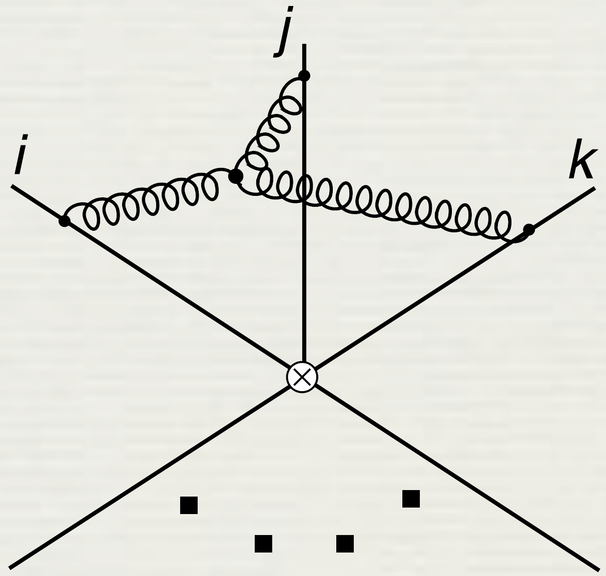
$$\mathbf{\Gamma}_s(\{\underline{\beta}\},\mu) = -\sum_{(i,j)} \frac{\boldsymbol{T}_i\cdot\boldsymbol{T}_j}{2}\,\gamma_{\text{cusp}}(\alpha_s)\,\beta_{ij} + \sum_i \gamma_s^i(\alpha_s)$$

- This form was confirmed at two loops by showing that diagrams connecting three parton legs vanish

  Mert Aybat, Dixon, Sterman 2006

- Also holds for three-loop fermionic contributions

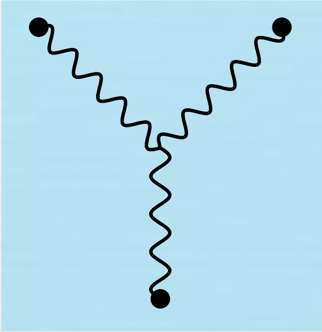
  Dixon 2009



Order-by-order analysis

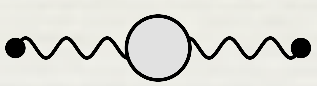
- One loop (recall $\sum_{(i,j)} \boldsymbol{T}_i \cdot \boldsymbol{T}_j = -\sum_i \boldsymbol{T}_i^2 = -\sum_i C_i$)
  - one leg: $\boldsymbol{T}_i^2 = C_i$

  - two legs: $\boldsymbol{T}_i \cdot \boldsymbol{T}_j$
- Two loops



  - one leg: $-i f^{abc}\, \boldsymbol{T}_i^a \boldsymbol{T}_i^b \boldsymbol{T}_i^c = \dfrac{C_A C_i}{2}$
  - two legs: $-i f^{abc}\, \boldsymbol{T}_i^a \boldsymbol{T}_i^b \boldsymbol{T}_j^c = \dfrac{C_A}{2} \boldsymbol{T}_i \cdot \boldsymbol{T}_j$

(only new structure)



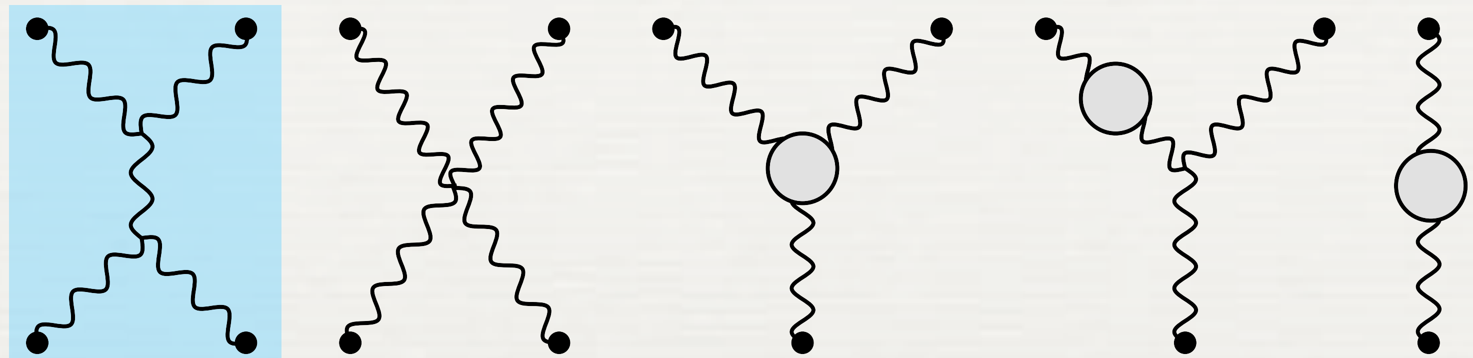
  - three legs: $-i f^{abc}\, \boldsymbol{T}_i^a \boldsymbol{T}_j^b \boldsymbol{T}_k^c$

$\Rightarrow$ vanishes, since no antisymmetric momentum structure in i,j,k consistent with soft-collinear factorization exists!

explains cancellations observed in:
Mert Aybat, Dixon, Sterman 2006; Dixon 2009

Three-loop order

- Single webs:



(only new structure)

- Six new structures consistent with non-abelian exponentiation exist, two of which are compatible with soft-collinear factorization:

$$\boldsymbol{\Delta\Gamma}_3(\{\underline{p}\},\mu) = -\frac{\bar{f}_1(\alpha_s)}{4} \sum_{(i,j,k,l)} f^{ade} f^{bce}\, \boldsymbol{T}_i^a\, \boldsymbol{T}_j^b\, \boldsymbol{T}_k^c\, \boldsymbol{T}_l^d \, \ln \frac{(-s_{ij})(-s_{kl})}{(-s_{ik})(-s_{jl})}$$

$$- \bar{f}_2(\alpha_s) \sum_{(i,j,k)} f^{ade} f^{bce} \left(\boldsymbol{T}_i^a\, \boldsymbol{T}_i^b\right)_+ \boldsymbol{T}_j^c\, \boldsymbol{T}_k^d \,,$$

more generally, arbitrary odd function of conformal cross ratio

Three-loop order

- Neither of these is compatible with collinear limits: the splitting function would depend on colors and momenta of the additional partons
- Consider, e.g., the second term:

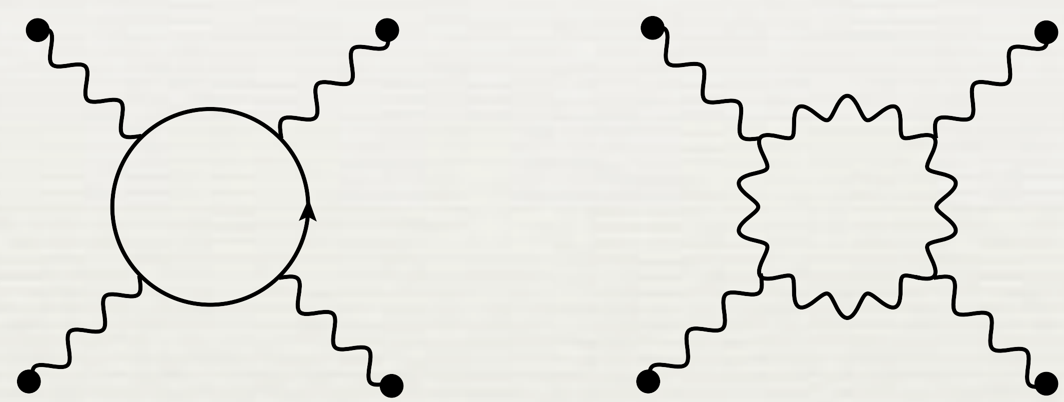
$$\boldsymbol{\Delta\Gamma}_{\mathrm{Sp}}(\{p_1,p_2\},\mu)\big|_{\bar{f}_2(\alpha_s)} = 2f^{ade}f^{bce}\Big[\big(\boldsymbol{T}_1^a\boldsymbol{T}_1^b\big)_+\big(\boldsymbol{T}_2^c\boldsymbol{T}_2^d\big)_+ - \sum_{i\neq 1,2}\big(\boldsymbol{T}_1^a\boldsymbol{T}_2^b+\boldsymbol{T}_2^a\boldsymbol{T}_1^b\big)\big(\boldsymbol{T}_i^c\boldsymbol{T}_i^d\big)_+\Big]$$

$$\boldsymbol{\Delta\Gamma}_{\mathrm{Sp}}(\{p_1,p_2\},\mu)\big|_{\bar{f}_1(\alpha_s)} = f^{ade}f^{bce}\sum_{(i,j)\neq 1,2}\big(\boldsymbol{T}_1^a\boldsymbol{T}_2^b+\boldsymbol{T}_2^a\boldsymbol{T}_1^b\big)\boldsymbol{T}_i^c\boldsymbol{T}_j^d\ln\frac{\mu^2}{-s_{ij}} + \ \dots$$

dependence on color invariants and momenta of additional partons (i≠1,2)

Four-loops and beyond

- Interesting new webs involving higher Casimir invariants first arise at four loops



$$d_F^{abcd}\,\boldsymbol{T}_i^a\,\boldsymbol{T}_j^b\,\boldsymbol{T}_k^c\,\boldsymbol{T}_l^d = d_F^{abcd}\left(\boldsymbol{T}_i^a\,\boldsymbol{T}_j^b\,\boldsymbol{T}_k^c\,\boldsymbol{T}_l^d\right)_+$$

$$d_R^{a_1 a_2 \ldots a_n} = \mathrm{tr}\Big[\left(\boldsymbol{T}_R^{a_1}\,\boldsymbol{T}_R^{a_2}\ldots\boldsymbol{T}_R^{a_n}\right)_+\Big]$$

- One linear combination of such terms would be compatible with soft-collinear factorization, but does not have the correct collinear limit

Casimir scaling

- Applied to the two-jet case (form factors), our formula thus implies Casimir scaling of the cusp anomalous dimension:

$$\frac{\Gamma^q_{\text{cusp}}(\alpha_s)}{C_F} = \frac{\Gamma^g_{\text{cusp}}(\alpha_s)}{C_A} = \gamma_{\text{cusp}}(\alpha_s)$$

- Checked explicitly at three loops Moch, Vermaseren, Vogt 2004
- But contradicts expectations from AdS/CFT correspondence (high-spin operators in strong-coupling limit) Armoni 2006, Alday, Maldacena 2007
- Presumably not a real conflict ...

Wanted: 3- and 4-loop checks

- Full three-loop 4-jet amplitudes in N=4 super Yang-Mills theory were expressed in terms of small number of scalar integrals
  Bern et al. 2008
- Once these can be calculated, this will provide stringent test of our arguments (note recent calculation of three-loop form-factor integrals)
  Baikov et al. 2009;
  Heinrich, Huber, Kosower, Smirnov 2009
- Calculation of four-loop cusp anomalous dimension would provide non-trivial test of Casimir scaling, which is then no longer guaranteed by non-abelian exponentiation

... and applications

Generalizations

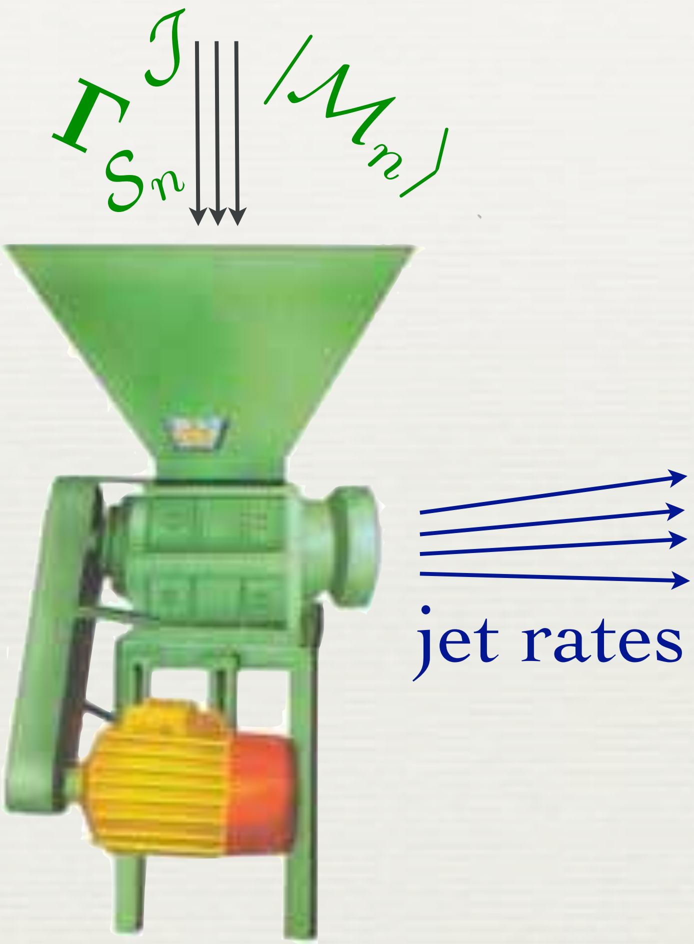
- Have established conjecture for anomalous-dimension matrix up to three loops (four loops for cusp-log part)
  - sufficient for NNNLL resummations (good enough in practice)
  - all-orders proof should be possible
- Extensions to massive partons should be possible, generalizing existing methods

Catani, Dittmaier, Trocsanyi 2000
Becher, Melnikov 2007
Mitov, Moch 2007

Main phenomenological application

- Beyond LL resummation of Sudakov logarithms:
  - hard functions known from fixed-order results for on-shell amplitudes (use matrix-element generators to obtain results for arbitrary n)
  - new unitarity methods allow calculation of one-loop amplitudes with many legs ($\rightarrow$ NNLL resummation)
  - need to calculate soft and jet functions for given observable
  - solve RG equations

Automatization



- in the longer term, this will hopefully lead to automated higher-log resummations for jet rates
- goes beyond parton showers, which are only accurate at LL, even after matching
- predicts jets, not individual partons

Conclusions

- SCET provides transparent way to separate contributions from different mass scales (hard, collinear, soft), and efficient method to resum associated logarithms by RG evolution
- Finally on track to analyse non-trivial, unsolved problems, such as higher-log resummation for n-jet production at LHC
- Most non-trivial task (evolution of hard matching coefficient) has been completed
- Solves old problem of understanding IR divergences of QCD scattering amplitudes