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18.781 pset 7
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3-5 1. $F = \begin{bmatrix} 7 & 25/2 \\ 25/2 & 23 \end{bmatrix}$. Let $M = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$; then $M^tFM = \begin{bmatrix} 7 & -3/2 \\ -3/2 & 1 \end{bmatrix}$, representing $7x^2 - 3xy + y^2$. Then we transform it to $x^2 + 3xy + 7y^2$, and then to $x^2 + 3xy + 5y^2$, which is reduced.

3-5 3. $\begin{bmatrix} x & y \\ u & v \end{bmatrix} \in \Gamma$ implies $xv - yu = 1$, which by the extended Euclidean algorithm implies $(x, y) = 1$.

3-5 4. In one direction, the quadratic form $nx^2 + bxy + cy^2$ for any b and c represents n as $(1, 0)$. In the other direction,

3-5 5.

3-5 9. Let $n = ax^2 + bxy + cy^2$. Then $b^2 - 4ac \pmod{|d|}$, so $4an \equiv 4a^2x^2 + 4abxy + 4acy^2 \equiv 4a^2x^2 + 4abxy + b^2y^2 \equiv (2ax + by)^2 \pmod{|d|}$.

3-6 1. Consider integers less than 13. These can only be made by combining 1, 4, and 9; in particular 6, 7, 8, and 9 are not sums of squares.

3-6 2. We want a number composed of as many primes of the form $4k + 1$ as possible, and the smallest such primes. $R(n)$ is 4 for each prime, plus another 4 for each unique prime. We can limit our search to about 5, 13, and 17, for anything else would be too big; by trying a few possibilities, we find that $R(5^2 \cdot 13 = 325) = 24$, and we can go no higher.

3-6 4. This problem looks silly.

3-6 6. If there are k solutions (x, y) to $x^2 + y^2 = n$ over the positive integers where $(x, y) = 1$, then $(x, y), (x, -y), (-x, y), (-x, -y)$ are all solutions to $x^2 + y^2 = n$ over the integers, and are distinct because neither x nor y can be zero. So the number of solutions $r(n) = 4k$. Therefore $k = r(n)/4$.

3-6 8. For integers p, q, r, s, where $(p, q) = (r, s) = 1$, write $n = \frac{p^2}{q^2} + \frac{r^2}{s^2} = \frac{p^2s^2 + r^2q^2}{q^2s^2}$. Reduce this fracction to simplest form as follows: suppose it is not. Then there were some factor k in the denominator, it must be a factor of either q or s. WLOG let it be a factor of q. Then if $k | p^2s^2 + r^2q^2$, it is also the case that $k | p^2s^2$. But we know $k \not| p^2$ because $(p, q) = 1$, so $k | s^2$ and $k | s$, so k is a factor of both q and s. Therefore we can cancel k and rewrite using $q' = q/k$ and $s' = s/k$. Ultimately, this fraction needs to be an integer, so the denominator is 1. Therefore, n is the sum of the squares $(ps)^2 + (rq)^2$.