18.701 Lecture Notes

2007-09-05

General Linear Group

$$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1a + 3c & 1b + 3d \\ 2a + 4c & 2b + 4c \end{pmatrix}$$

- Associative: $(AB)C = A(BC)$
- Idenitity I $= IA = A, AI = A$
- Inverse matrix A^{-1}
  $\frac{1}{ab-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
  A is invertible iff $\det A \neq 0$

GL_n = general linear group = $\{n \times n$ matrices, invertible $\}$
$GL_n(\mathbb{R})$ with real entries.

Elementary Matrices

1. $\begin{pmatrix} 1 & \\ x & 1 \end{pmatrix}, \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$
2. $\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$
3. $\begin{pmatrix} y & \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ & y \end{pmatrix}$

$$\begin{pmatrix} 1 & \\ x & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ xa + c & xb + d \end{pmatrix}$$

$$\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$$

$$\begin{pmatrix} y & \\ & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ya & yb \\ c & d \end{pmatrix}$$

Thm. *Every element of GL_n is a product of elementary matrices*

aka "The elementary matrices generate the general linear group"

$$\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \\ -3 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 4 \\ 0 & -10 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \\ & \frac{-1}{10} \end{pmatrix} \to \begin{pmatrix} 1 & 4 \\ & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -4 \\ & 1 \end{pmatrix} \to \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$$

Symmetric Group

U, V sets $\phi : U \to V$

- ϕ injective iff $u \neq v \in U \Rightarrow \phi(u) \neq \phi(v)$
- ϕ surjective iff $v \in V \Rightarrow \exists u \in U$ s.t. $\phi(u) = v$
- ϕ bijective iff ϕ injective and surjective
- ϕ bijective iff $\exists \phi^{-1}$ s.t. $\phi\phi^{-1} = id_V$ and $\phi^{-1}\phi = id_U$
- if U, V finite and $|U| = |V|$ then ϕ is bijective $\iff$ ϕ injective $\iff$ ϕ surjective

Defn. A permutation of a set U is a bijective map $b : U \to U$

$ifq, p : U \to U$ a permutation, qp is a permutation.

- Multiplication is associative
- There is an identity

- Every permutation has an inverse (bijections)

Defn. The Symmetric Group S_n is $\{$permutations of $(1, 2, 3, \ldots, n)\}$

$|S_n| = n!$

Defn. A group G is a set with a law of composition that

- Is associative
- Has an idenity
- Every element has an inverse

Cycle notation

A 4-cycle:

$(3, 1, 5, 2)$

"Send 3 to 1, to 5, 5 to 2, and 2 to 3"

i	1	2	3	4	5	6
$p(i)$	4	3	1	2	6	5

$p = (1423)(56)$

Say $q = (36542)(1)$.

$qp = (12643)(5)$