Nets vs hierarchies for hard optimization problems

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outline

1. separable states and operator norms
2. approximating the set of separable states
3. approximating general operator norms
4. the simple case of the simplex

entanglement and optimization

Definition: ρ is separable (i.e. not entangled) if it can be written as

$$\rho = \Sigma_i \; p_i \; |v_i\rangle\langle v_i| \otimes |w_i\rangle\langle w_i|$$

probability distribution

unit vectors

$$\text{Sep} = \text{conv}\{|v_i\rangle\langle v_i| \otimes |w_i\rangle\langle w_i|\}$$
$$= \text{conv}\{\rho\otimes\sigma\}$$
$$=$$

Weak membership problem: Given ρ and the promise that $\rho \in \text{Sep}$ or ρ is far from Sep, determine which is the case.

Optimization: $h_{\text{Sep}}(M) := \max \{ \text{tr}[M\rho] : \rho \in \text{Sep} \}$

operator norms

X:A->B

$\|X\|_{A\text{->}B} = \sup \|Xa\|_B \,/\, \|a\|_A$

operator norm

Examples

$l_2 \to l_2$	largest singular value
$l_\infty \to l_1$	MAX-CUT = $\max\{\langle \mathrm{vec}(X), a\otimes b\rangle : \|a\|_\infty, \|b\|_\infty \le 1\}$
$l_1 \to l_\infty$	$\max_{i,j} \lvert X_{i,j}\rvert = \max\{\langle \mathrm{vec}(X), a\otimes b\rangle : \|a\|_1, \|b\|_1 \le 1\}$
S_1 -> S_1 of $X\otimes \mathrm{id}$	channel distinguishability (cb norm, diamond norm)
S_1 -> S_p	max output p-norm, min output Rènyi-p entropy
$l_2 \to l_4$	hypercontractivity, small-set expansion
$S_1 \to S_\infty$	$h_{Sep} = \max\{ \langle \mathrm{Choi}(X), a\otimes b\rangle : \|a\|_{S_1}, \|b\|_{S_1} \le 1 \}$

complexity of h_{Sep}

$h_{Sep}(M) \pm 0.1\ ||M||_{2\to 2}$ at least as hard as
- planted clique [Brubaker, Vempala '09]
- 3-SAT[$\log^2(n)$ / polyloglog(n)] [H, Montanaro '10]

$h_{Sep}(M) \pm 100\ h_{Sep}(M)$ at least as hard as
- small-set expansion [Barak, Brandão, H, Kelner, Steurer, Zhou '12]

$h_{Sep}(M) \pm ||M||_{2\to 2}$ / poly(n) at least as hard as
- 3-SAT[n] [Gurvits '03], [Le Gall, Nakagawa, Nishimura '12]

complexity of $l_2 \rightarrow l_4$ norm

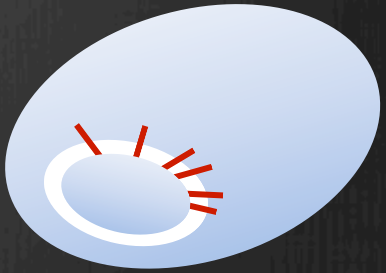
Unique Games (UG):

Given a system of linear equations: $x_i - x_j = a_{ij} \mod k$.
Determine whether $\geq 1-\epsilon$ or $\leq \epsilon$ fraction are satisfiable.

Small-Set Expansion (SSE):

Is the minimum expansion of a set with $\leq \delta n$ vertices $\geq 1-\epsilon$ or $\leq \epsilon$?

UG ≈ SSE ≤ 2->4



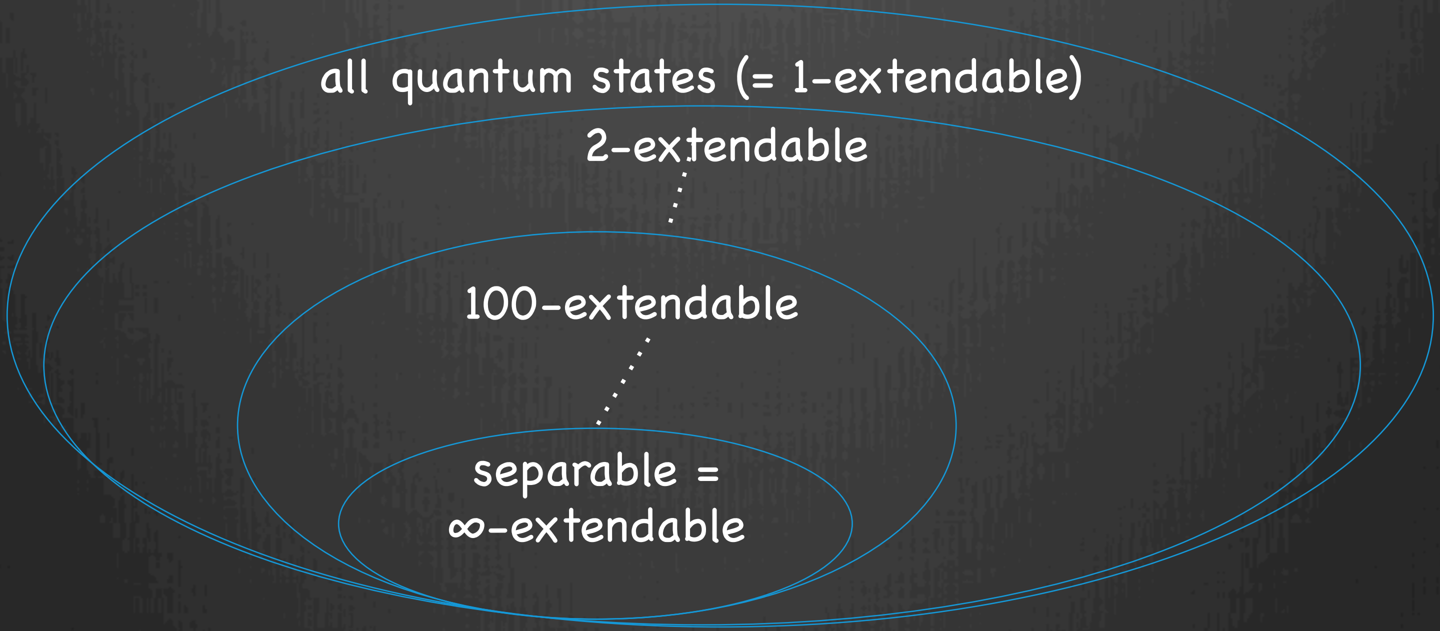
G = normalized adjacency matrix
P_λ = largest projector s.t. $G \geq \lambda P$

Theorem:

All sets of volume $\leq \delta$ have expansion $\geq 1 - \lambda^{O(1)}$
iff
$\|P_\lambda\|_{2\to 4} \leq n^{-1/4}/\delta^{O(1)}$

A hierachy of tests for entanglement

Definition: ρ^{AB} is k-extendable if there exists an extension $\rho^{AB_1 \ldots B_k}$ with $\rho^{AB} = \rho^{AB_i}$ for each i.



Algorithms: Can search/optimize over k-extendable states in time $n^{O(k)}$.

Question: How close are k-extendable states to separable states?

SDP hierarchies for h_{Sep}

$\mathrm{Sep}(n,m) = \mathrm{conv}\{\rho_1 \otimes \dots \otimes \rho_m : \rho_m \in D_n\}$
$\mathrm{SepSym}(n,m) = \mathrm{conv}\{\rho^{\otimes m} : \rho \in D_n\}$

bipartite

doesn't match hardness

Thm: If $M = \sum_i A_i \otimes B_i$ with $\sum_i |B_i| \leq I$, each $|A_i| \leq I$, then
$h_{Sep(n,2)}(M) \leq h_{k\text{-}ext}(M) \leq h_{Sep(n,2)}(M) + c\,(\log(n)/k)^{1/2}$

[Brandão, Christandl, Yard '10], [Yang '06], [Brandão, H '12], [Li, Winter '12]

multipartite

$$M = \sum_{i_1,\dots,i_m} c_{i_1,\dots,i_m} A^{(1)}_{i_1} \otimes \dots \otimes A^{(m)}_{i_m} \qquad \sum_i |A^{(j)}_i| \leq I \qquad |c_{i_1,\dots,i_m}| \leq 1$$

Thm:
ε-approx to $h_{SepSym(n,m)}(M)$ in time $\exp(m^2 \log^2(n)/\varepsilon^2)$.
ε-approx to $h_{Sep(n,m)}(M)$ in time $\exp(m^3 \log^2(n)/\varepsilon^2)$.

[Brandão, H '12], [Li, Smith '14]

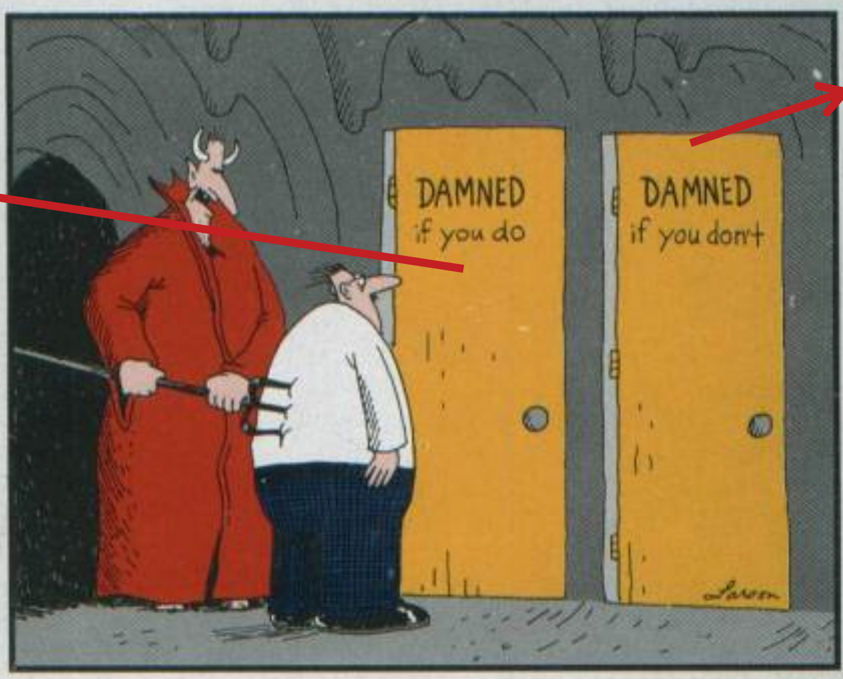
≈matches Chen-Drucker hardness

proof intuition

Measure extended state and get outcomes $p(a,b_1,\ldots,b_k)$.
Possible because of 1-LOCC form of M.

<u>case 1</u>

$p(a,b_1) \approx p(a) \cdot p(b_1)$



"C'mon, c'mon — it's either one or the other."

<u>case 2</u>

$p(a, b_2 \mid b_1)$

has less mutual information

questions

- Run-time $\exp(c \log^2(n) / \varepsilon^2)$ appears in both
    - Algorithm for M in 1-LOCC
    - Hardness for M in SEP.

Why? Can we bridge the gap?

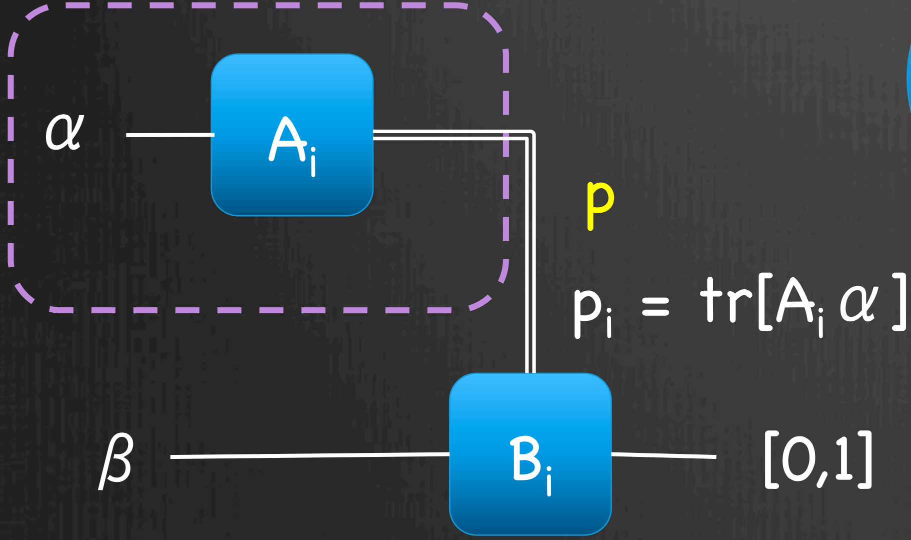
- Can we find multiplicative approximations, or otherwise use these approaches for SSE?

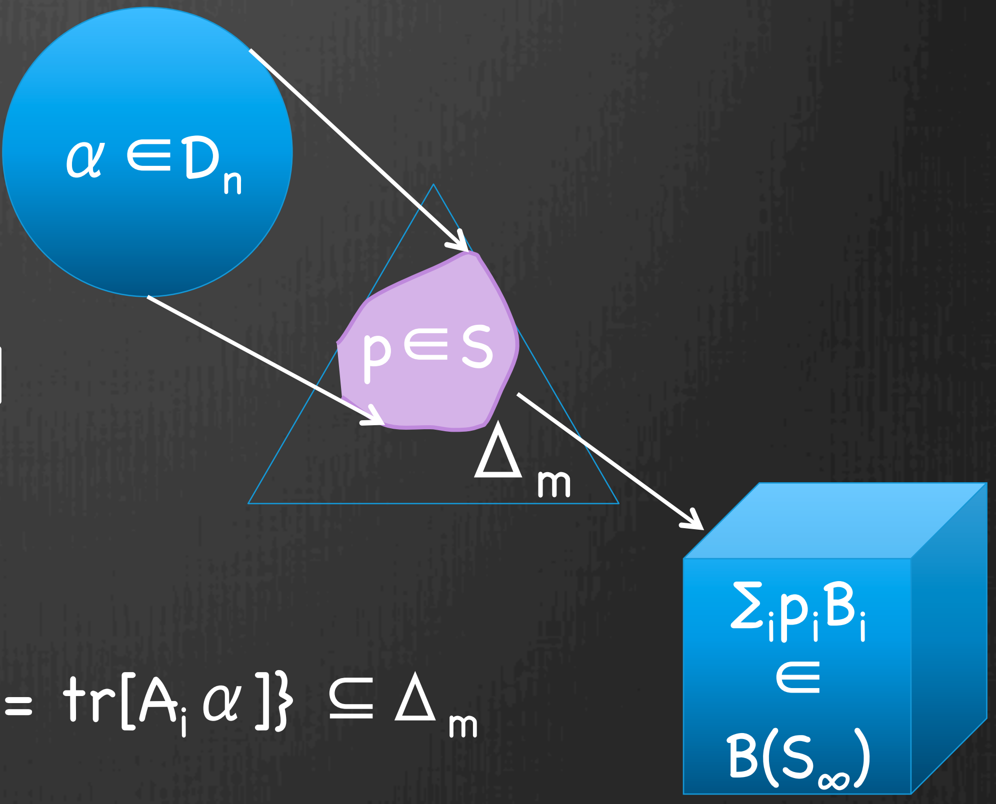
net-based algorithms

$M = \Sigma_{i\in[m]} A_i \otimes B_i$ with $\Sigma_i A_i \leq I$, each $|B_i| \leq I$, $A_i \geq 0$

Hierarchies estimate $h_{Sep}(M) \pm \varepsilon$ in time $\exp(\log^2(n)/\varepsilon^2)$

$$h_{Sep}(M) = \max_{\alpha,\beta} \mathrm{tr}[M(\alpha \otimes \beta)] = \max_{p\in S} \|p\|_B$$



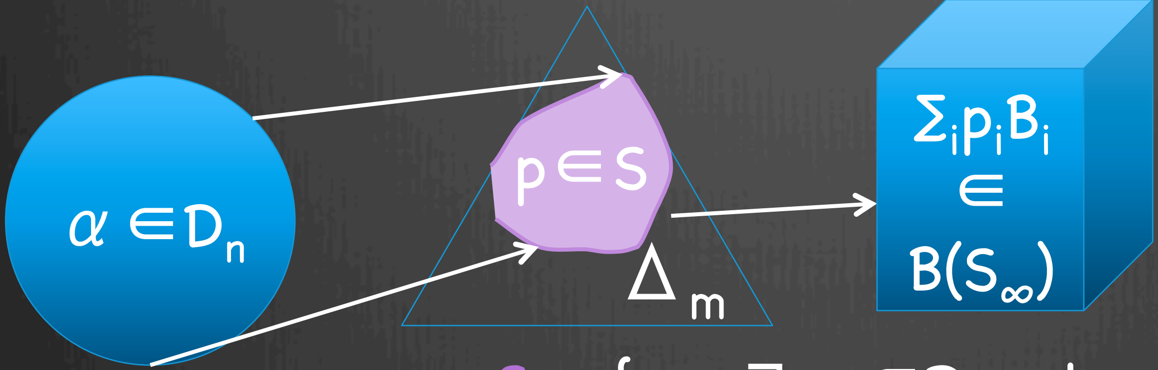


$S = \{p : \exists\, \alpha \in D_n \text{ s.t. } p_i = \mathrm{tr}[A_i \alpha]\} \subseteq \Delta_m$

$\|x\|_B = \|\Sigma_i x_i B_i\|_{2\text{->}2}$

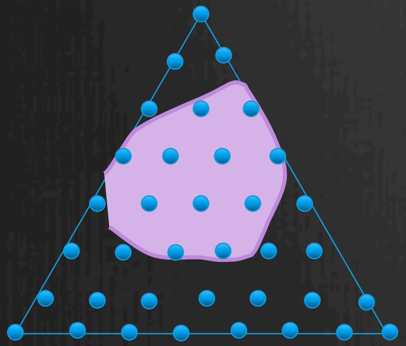
net-based algorithms

$h_{Sep}(M) = \max_{\alpha,\beta} \mathrm{tr}[M(\alpha \otimes \beta)] = \max_{p \in S} \|p\|_B$



$\|x\|_B = \|\Sigma_i\ x_i\ B_i\|_{2\text{->}2}$

$S = \{p : \exists\ \alpha \in D_n \text{ s.t. } p_i = \mathrm{tr}[A_i \alpha]\}$



Lemma: $\forall p \in \Delta_m$ $\exists q$ k-sparse (i.e. $\in \mathbb{Z}^m/k$) s.t. $\|p-q\|_B \leq c(\log(n)/k)^{1/2}$.

Pf: matrix Chernoff [Ahlswede-Winter]

Algorithm:

Enumerate over k-sparse q

- check whether $\exists p \in S$, $\|p-q\|_B \leq \varepsilon$
- if so, compute $\|q\|_B$

Performance

$k \simeq \log(n)/\varepsilon^2$, m=poly(n)

run-time

$O(m^k) = \exp(\log^2(n)/\varepsilon^2)$

nets for Banach spaces

X:A->B

$||X||_{A\to B}$ = sup $||Xa||_B$ / $||a||_A$ **operator norm**

$||X||_{A\to C\to B}$ = min $\{||Z||_{A\to C}\ ||Y||_{C\to B}$: X=YZ$\}$ **factorization norm**

Let A,B be arbitrary. $C = l_1^m$

Only changes are sparsification (cannot assume m≤poly(n))

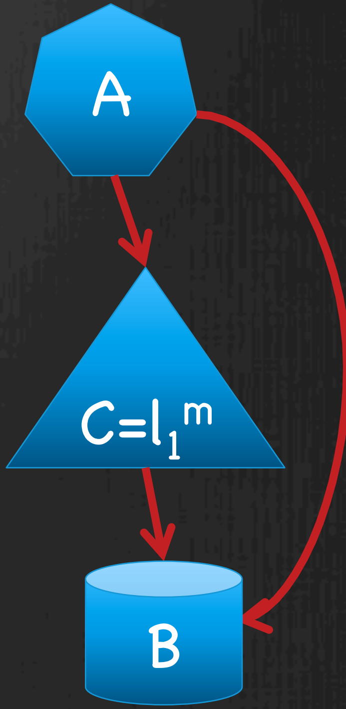
and operator Chernoff for B.

Type-2 constant: $T_2(B)$ is smallest λ such that

$$\mathbb{E}_{\epsilon_1,\ldots,\epsilon_n\in\{\pm1\}}\left\|\sum_{1=1}^{n}\epsilon_i Z_i\right\|_B^2 \leq \lambda^2\sum_{1=1}^{n}\|Z_i\|_B^2$$

result: $\|X\|_{A\to B} \pm \epsilon\|X\|_{A\to\ell_1^m\to B}$

estimated in time $\exp(T_2(B)^2 \log(m)/ \varepsilon^2)$



applications

$S_1 \rightarrow S_p$ norms of entanglement-breaking channels

$N(\rho) = \Sigma_i \operatorname{tr}[A_i \rho] B_i$, where $\Sigma_i A_i = I$, $\|B_i\|_1 = 1$.

Can estimate $\|N\|_{1\rightarrow p} \pm \varepsilon$ in time $n^{O(c)}$ where

$c = p/\varepsilon^2$ for $p \geq 2$

$c = (p/\varepsilon^p)^{1/(p-1)}$ for $1<p<2$

(uses bounds on $T_2(S_p)$ from [Ball-Carlen-Lieb '94]

low-rank measurements:

$h_{Sep}(\Sigma_i A_i \otimes B_i) \pm \varepsilon$ for

$\Sigma_i |A_i| = 1$, $\|B_i\|_\infty \leq 1$, rank $B_i \leq r$

in time $n^{O(r/\varepsilon^2)}$

$l_2 \rightarrow l_p$ for even $p \geq 4$

$$\|X\|_{2\to p}^p \pm \epsilon \|X\|_{2\to 2}^2 \|X\|_{2\to\infty}^{p-2}$$

in time $n^{O(p/\varepsilon^2)}$

Multipartite versions of 1-LOCC norm too [cf. Li-Smith '14]

ε -nets vs. SoS

Problem	ε -nets	SoS/info theory
$\max_{p\in\Delta} p^TAp$	BK ’02, KLP ‘06	DF ’80 BK ’02, KLP ‘06
approx Nash	LMM ’03	HNW ‘16
free games	AIM ‘14	BH ‘13
unique games	ABS ’10	BRS ‘11
small-set expansion	ABS ’10	BBHKSZ ’12
h_{Sep}	SW ‘11 BH ‘15	BCY ‘10 BH ’12 BKS ‘13

simplest version: polynomial optimization over the simplex

$\Delta_n = \{p \in \mathbb{R}^n : p \geq 0, \Sigma_i\, p_i = 1\}$

Given homogenous degree-d poly $f(p_1, \ldots, p_n)$, find $\max_p f(p)$.

NP-complete: given graph G with clique number α, $\max_p p^T A p = 1 - 1/\alpha$. [Motzkin-Strauss, '65]

Approximation algorithms

- Net: Enumerate over all points in $\Delta_n(k) := \Delta_n \cap \mathbb{Z}^n/k$.
- Hierarchy: min λ s.t. $(\Sigma_i\, p_i)^k\, (\lambda\, (\Sigma_i\, p_i)^d - f(p))$ has all nonnegative coefficients.

Thm: Each gives error $\leq (\max_p f(p) - \min_p f(p)) \exp(d) / k$ in time $n^{O(k)}$. [de Klerk, Laurent, Parrilo, '06]

sum-of-squares (SoS) proofs

Axioms:

$g_1(x) \geq 0$
$\vdots$
$g_m(x) \geq 0$ $\xrightarrow{\text{derive}}$ $f(x) \leq \lambda$

Rules:

1. polynomial operations
2. intermediate polys have deg $\leq k$
3. [optional: changes LP to SDP]

$r(x)^2 \geq 0$ for any polynomial $r(x)$

"Positivestellensatz" [Stengel '74]

hierarchies & SoS proofs

Given axioms: $\Sigma_i\ p_i = 1$ and $p_i \geq 0$
prove that $\lambda - f(p) \geq 0$.

Previous strategy:
$\lambda\,(\Sigma_i\ p_i)^d - f(p) = (\Sigma_i\ p_i)^k\ (\lambda\,(\Sigma_i\ p_i)^d - f(p) \geq 0$

difference is divisible
by $1 - \Sigma_i\ p_i$

LHS is nonnegative sum
of products of p_i

Dual is equivalent to net enumeration for modified objective function.
[Bomze, de Klerk '02] [de Klerk, Laurent, Sun '14]

k-extendable hierarchy

For a deg-d homogenous poly f(p), define $\mathrm{vec}(f) \in (R^n)^{\otimes d}$ to be the symmetric tensor such that $f(x) = \langle \mathrm{vec}(f), x^{\otimes d} \rangle$.

Then $\max_p f(p) = h_K(\mathrm{vec}(f))$ for
$K = \mathrm{conv}\{p^{\otimes d} : p \in \Delta_n\}$
$h_K(y) := \max_{x \in K} \langle x, y \rangle$

relaxation:
$q \in \Delta_{n^{d+k}}$ symmetric (aka "exchangeable")
$\pi = q^{(1,2,..,d)}$

convergence: [Diaconis, Freedman '80]
$\mathrm{dist}(\pi, \mathrm{conv}\{p^{\otimes d}\}) \leq O(d^2/k)$
$\rightarrow$ error $\|\mathrm{vec}(f)\|_\infty / k$ in time $n^{O(k)}$

Nash equilibria

Non-cooperative games:

Players choose strategies $p^A \in \Delta_m$, $p^B \in \Delta_n$.

Receive values $\langle V_A, p^A \otimes p^B \rangle$ and $\langle V_B, p^A \otimes p^B \rangle$.

Nash equilibrium: neither player can improve own value

ε -approximate Nash: cannot improve value by $> \varepsilon$

Correlated equilibria:

Players follow joint strategy $p^{AB} \in \Delta_{mn}$.

Receive values $\langle V_A, p^{AB} \rangle$ and $\langle V_B, p^{AB} \rangle$.

Cannot improve value by unilateral change.

- Can find in poly(m,n) time with LP.
- Nash equilibrium = correlated equilibrum with $p = p^A \otimes p^B$

finding (approximate) Nash eq

Known complexity:
Finding exact Nash eq. is PPAD complete.
Optimizing over exact Nash eq is NP-complete.

Algorithm for ε-approx Nash in time $\exp(\log(m)\log(n)/\varepsilon^2)$ based on enumerating over nets for Δ_m, Δ_n.
Planted clique and 3-SAT[$\log^2(n)$] reduce to optimizing over ε-approx Nash.

[Lipton, Markakis, Mehta '03], [Hazan-Krauthgamer '11], [Braverman, Ko, Weinstein '14]

New result [HNW16]: Another algorithm for finding ε-approximate Nash with the same run-time.

(uses k-extendable distributions)

algorithm for approx Nash

Search over $p^{AB_1\dots B_k} \in \Delta_{mn^k}$
such that the A:B_i marginal is a correlated equilibrium conditioned on any values for B_1, ..., B_{i-1}.

LP, so runs in time poly(mn^k)

Claim: Most conditional distributions are ≈ product.

Proof:

$\log(m) \geq H(A) \geq I(A:B_1\dots B_k) = \Sigma_{1\leq i\leq k}\ I(A:B_i|B_{<i})$

$\mathbb{E}_i\ I(A:B_i|B_{<i}) \leq \log(m)/k =: \ \varepsilon^2$

$\therefore\ k = \log(m)/\varepsilon^2$ suffices.

open questions

- Application to unique games, small-set expansion, etc. Which norms are the right ones here?
- Tight hardness results, e.g. for h_{Sep}.
- Explain the coincidences!