1. (1) Prove what Jeffery called the “strongest wins principle”: if \( x, y \in \mathbb{Q} \) are such that \( |x|_p \neq |y|_p \), then \( |x + y|_p = \max(|x|_p, |y|_p) \).

   (2) Prove that multiplication and addition are continuous functions \( \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q} \) for the \( p \)-adic topology (i.e., the topology induced by the \( p \)-adic metric).

2. Recall that for \( a \in \mathbb{Q} \) and \( r \in \mathbb{R} \), Tristan defined the “open ball” \( B(a, r) := \{ x \in \mathbb{Q} : d_p(a, x) < r \} \) and the “closed ball” \( \overline{B}(a, r) := \{ x \in \mathbb{Q} : d_p(a, x) \leq r \} \).

   (1) Tristan said that “every point of \( B(a, r) \) is a center”. Formulate rigorously what this means and then prove it.

   (2) Show that for every \( r \), there exists \( r' \) such that \( B(a, r) = \overline{B}(a, r') \). Is it true that for every \( r' \), there exists \( r \) such that \( \overline{B}(a, r') = B(a, r) \)? Prove it or give a counterexample.

3. Suppose that \( p \) and \( \ell \) are two different primes. Show that the \( p \)-adic absolute value is not equivalent to the \( \ell \)-adic absolute value.

4. Consider the sequence of integers (written in base 10) \( 4, 44, 444, 4444, 44444, \ldots \)
Zawad showed in class that this is a Cauchy sequence for the 5-adic norm. Find, with proof, its limiting value.