MUSIC AS EMBODIED MATHEMATICS:
A STUDY OF A MUTUALLY INFORMING AFFINITY

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Abstract

The argument examined in this paper is that music—when approached through making and responding to coherent musical structures and facilitated by multiple, intuitively accessible representations—can become a learning context in which basic mathematical ideas can be elicited and perceived as relevant and important. Students' inquiry into the bases for their perceptions of musical coherence provides a path into the mathematics of ratio, proportion, fractions, and common multiples. In a similar manner, we conjecture that other topics in mathematics—patterns of change, transformations and invariants—might also expose, illuminate and account for more general organizing structures in music. Drawing on experience with 11-12 year old students working in a software music/math environment, we illustrate the role of multiple representations, multi-media, and the use of multiple sensory modalities in eliciting and developing students' initially implicit knowledge of music and its inherent mathematics.

"Music is the arithmetic of the soul, which
counts without being aware of it."
--Leibnitz--

INTRODUCTION

Interest in the mutual affinities between music and mathematics has had a long history—Plato, Aristotle, Pythagoras, Leibnitz, and more recently Hofstadter (1979), Rothstein (1995), Lerdahl and Jackendoff (1983), Tanay (1998), and others. But unlike these carefully crafted and in some cases formal theories, the connections we discuss are empirical and "cognitively real" in the sense that they seem naturally embedded in the structures that generate the perception and invention of musical coherence. These functional connections initially came to the surface as college students reflected on their own creative processes during composition projects facilitated by the text *Developing Musical Intuitions* and its computer music environment, *Impromptu* (Bamberger, 2000).

The initial design of Impromptu was not at all intended to introduce mathematical principles. Instead, the text and software were meant to support an alternative approach to college level music fundamentals instruction. The goal was, as the title *Developing Musical Intuitions* suggests, to provide an environment where, rather than giving up their intuitions, students could learn in the service of developing and better understanding them (Bamberger, 1996).

To this end, students begin with semi-structured melodic composition projects, go on to create percussion accompaniments to their melodies, and eventually more complex, multi-part compositions. To encourage students to reflect on these activities, they are asked to keep a log of their decision-making process while composing. These logs (which students turn in with their completed compositions) have constituted an empirical base for an initial study of intuitive musical knowledge and its development (Bamberger, 2003). Indeed, it was in analyzing musically novice students' accounts of their work-in-progress, particularly as they experimented with rhythmic possibilities, that we noticed mathematical relationships playing a role in their perception and composition of musical coherence.

It may seem unremarkable that the principal mathematics college students spontaneously put to work involved ratio, proportion, fractions, and common multiples. However, it turns out that these intuitively generated and perceived music/mathematical relationships are some of the important mathematical concepts that are found to be most problematic for middle school children (Confrey & Smith, 1995; Wilensky & Resnick, 1999; Thompson, 1996; Arnon et al, 2001. Thus, it seemed worth exploring if music, through the mediation of Impromptu, could help children understand and effectively use this apparently troublesome mathematics. Engaging both domains together might also enhance the children's appreciation and understanding of aesthetic relations shared by mathematics and music.

To explore these ideas, we carried out an informal experiment with a group of 6th grade children in a multi-cultural, mixed socio-economic public school setting. Working together with one of their two regular classroom teachers, we (JB) met with a group of six children once or twice a week for 45 minutes over a period of three months. Activities were drawn, in part, from projects in *Developing Musical Intuitions*, facilitated by Impromptu. In addition, as a way of confronting their work in this virtual world with the more directly sensory experiences of real-time action and perception, computer-based projects were coupled with singing and playing real instruments—primarily drums of various sorts.

Impromptu, Mathematics, and Alternative Representations

Before considering the students' work, we need to provide some background on Impromptu along with a bit of music theory (for those who are not already familiar with it), and the psychology of representation. Subsequent sections will show how these ideas are realized in the work of children. In working with Impromptu, there are two basic aspects that initially encourage students to make practical use of structures shared by music and mathematics. The first aspect is internal to the structure of music, particularly (but not only) how music organizes time. The second aspect is the way these musical structures are represented in Impromptu.

With regard to the first, the most direct connection lies in the fact that all the music with which we are most familiar consistently generates an underlying periodicity. Formally, this is called a *beat*—that is, what you "keep time to," tap your foot to, in listening to music. The underlying beat becomes a temporal *unit* as it marks off the continuous flow of time into discrete and regularly recurring events—the "counts" alluded to by Leibnitz in the quote fronting this paper. Further, most familiar music generates several levels of beats—a hierarchy of temporal periodicities. Beats at each level occur at different rates, but there is a consistent proportional relationship among them—usually 2:1 or 3:1.

These periodic and proportional relations are easily responded to in action—clapping, swaying, dancing, tapping your foot. In contrast, through history, temporal relations have shown themselves to be persistently problematic to represent. In this regard, the history of th evolution of music notation is particularly cogent. Beginning around the 9th century and up until the 13th –14th Centuries, notations had been kinds of gestural squiggles inserted above t words in religious texts to guide the singers in how to coordinate words and music. These graphic marks, called *neumes,* represented whole little motifs as shown in Figure 1 where neither pitch nor rhythm ("notes") was specifically indicated at all. Thus, singing the text fro this notation depended largely on singers knowing the melody already—that is, the notatior was essentially a mnemonic device. Staff notation for pitch developed relatively rapidly, but was only in the mid-16th Century that rhythm notation as we now know it finally emerged. is noteworthy that, partly as a function of the characteristics of temporal organization in mu: up to that time, a central issue had been recognizing (or constructing) the notion that an underlying beat could serve as a "unit" with which consistently to measure and thus to represent the varied temporal events that were to be performed.

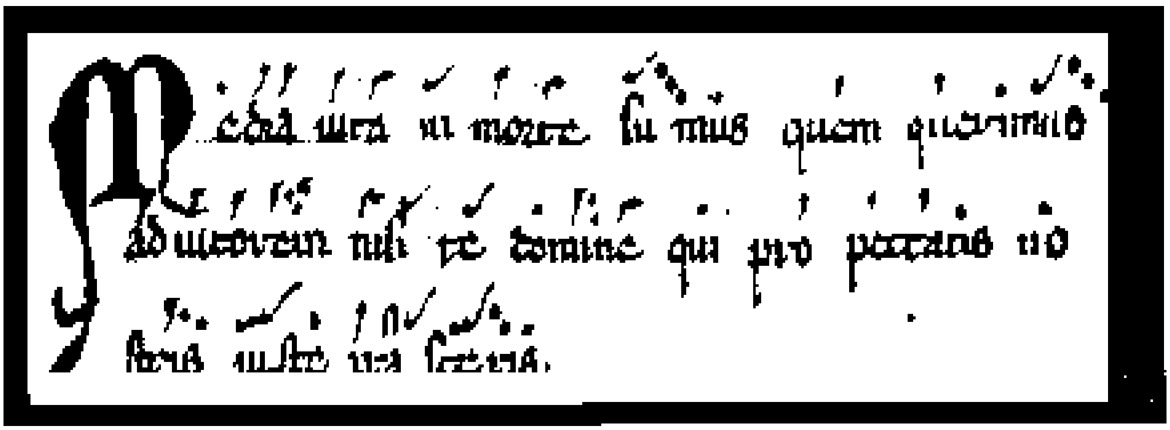

Figure 1. Gregorian Chant "neumes."

The issues arising around representations of continuous time and motion are not limite to music. Stated most generally: how do we transform the elusiveness of actions that take pl: continuously through time, into representations that hold still to be looked at and upon wh to reflect. Christopher Hasty in his book, "Meter as Rhythm" puts it this way:

> "... how shall we account for those attributes of rhythm that point to the particularity an spontaneity of aesthetic experience as it is happening? To take measurements or to analy and compare patterns we must arrest the flow of music and seek quantitative representations of musical events…. To the extent we find it comprehensible, music is organized; but this is an organization that is communicated in process and cannot be captured or held fast." (Hasty, 1999: 4)[1]

[1] Time: First, does it belong to the class of things that exist or to that of things that do not exist? Then secondly, what is its nature? if a divisible thing is to exist, it is necessary that, when it exists, all or some of its parts must exist. But of time some parts have been, while others have to be, and no part of it *is*, though it is divisible. For what is "now" is not a part: a part is a measure of the whole, which must be made up of parts. Time, on the other hand, is not held to be made up of "nows."

Aristotle, Physics, Book IV, Chapter 10.

Instead of finessing these enigmas, we made an effort in designing Impromptu to confro them. In particular, by invoking multiple representations, we tried to make explicit the complex nature of transformations involved in moving between experienced action and sta representations. Indeed, as we will illustrate, in the process of coming to understand and use the impromptu representations, users' productive confusions have led them to discover interesting and surprising aspects of temporal phenomena.

Impromptu's Temporal Representations

A. Graphical represenations

Figure 2 shows the Impromptu graphics left behind when one of the synthesizer drums plays just the rhythm, the varied "durations," of the simple tune, Hot Cross Buns. The representation captures only the information available in clapping the tune, without singing it.

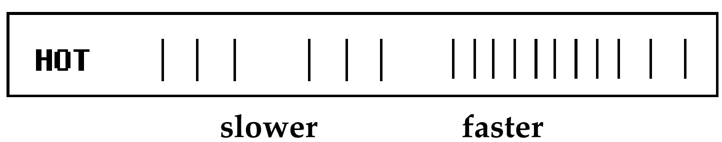

Figure 2. A representation of the rhythmic structure of "Hot Cross Buns." Spaces between lir show the relative durations of beats.

The unequally spaced vertical lines show a spatial analog for varied durations. "Duratio here, refers to the time from the onset of one event (clap) to the onset of the next. Thus, in tl graphics, events that take up more time (go slower in action), also take up more space. Similarly, events that take up less time (go faster) take up less space. We chose this spatial representation for actions in time because it is easy to explain: It is like the actual trace you would leave behind if you "played" a rhythm with a pencil on paper, moving the pencil up and down in one place, while pulling the paper continuously from right to left. Moreover, t representation is essentially borrowed from drawings children (even adults sometimes) mak spontaneously when asked to: “invent a way of putting on paper what you just clapped so someone else could clap it (Bamberger, 1995).

The top row of Figure 3 shows the Impromptu graphics for the rhythm of “Hot” and below it the three levels of beats that are being generated by the varied durations of the tune—the *metric hierarchy*.

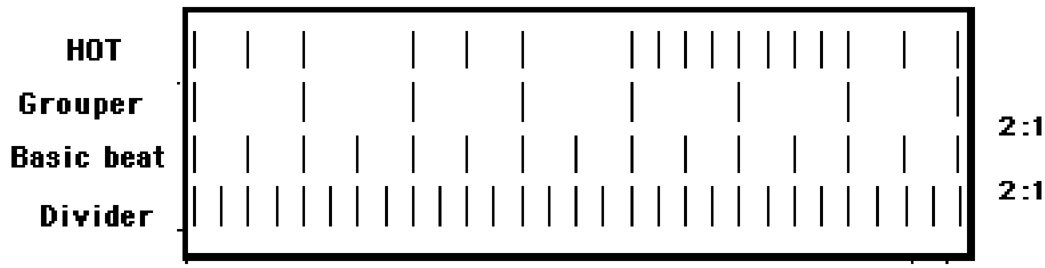

Figure 3. Beats in the metric hierarchy show constant proportions between levels.

To understand the graphics, sing the tune and just “keep time.” That is, instead of clapping the varied durations of the tune, just accompany the tune by clapping a steady beat that goes with it. Watching the graphics as you clap, you will probably find yourself clapping the mid-level beat shown in the graphics—the “basic beat” that “fits” most comfortably with the tune. If you sing the tune again, you can also clap a slower beat, called the “grouper,” which fits with the tune, as well. We call the slower beat the “grouper” beat because it groups the basic beat. If you can tap both these beats at once using two hands, you will find, as in the spatial graphics, that there are two basic beats for each grouper beat—a 2:1 relationship between these two rates. And, as in the graphics, you can also find and clap a third beat that goes twice as fast as the basic beat—that is, it divides the basic beat, again forming a 2:1 relationship. To summarize, three levels of beats are generated by the tune, and together they form its metric hierarchy.

If a piece of music, like Hot, generates a 2:1 relationship between basic beat and slower, grouper beat, it is said to be in *duple meter*. In contrast, if you listen to a common waltz tune such as Strauss’s “The Blue Danube,” you will find that the slower beat groups the basic beat into groups of three—a 3:1 relationship thus commonly called *triple meter*. Figure 4 shows a comparison between the proportional relations among beat levels in duple meter and typical triple meter.

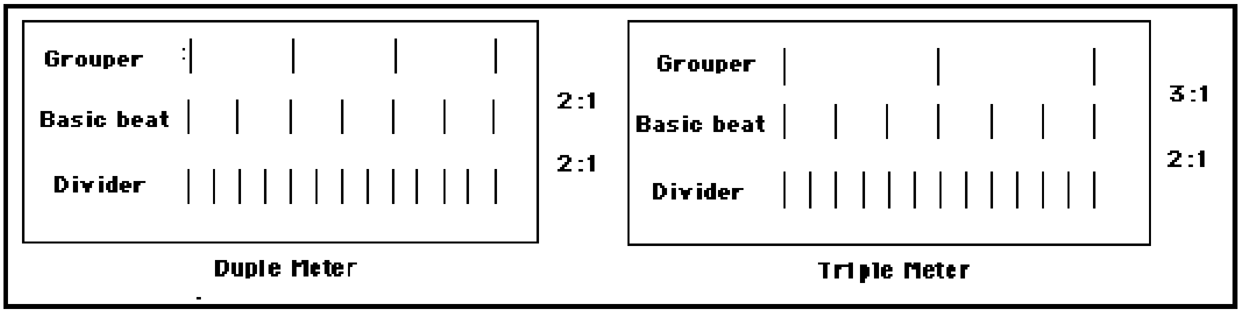

Figure 4. Duple and triple meter.

It should be evident from these examples that, unlike the arbitrary, outside fixed reference units typically used to measure and calculate in mathematics and science, (centimeter, calorie, gram), beats, as units of measure in music, are actually generated by the relations among events internal to the music itself. Beats are not seconds or any other "standard" unit of time. Instead, these are self-generated units that are used, in turn, as a kind of temporal ruler to measure the durationally varied events that are actually generating them—a nice example of self-reference.[2]

The periodicities at each level and the proportional relations among them arise because the relations among the varied durations of performed events are also primarily proportional. Figure 5 again shows the beat hierarchy as self-referencing units of measure of the proportional durations of Hot, this time with words to emphasize the relation between the surface-level of the melody and the metric hierarchy as temporal ruler:

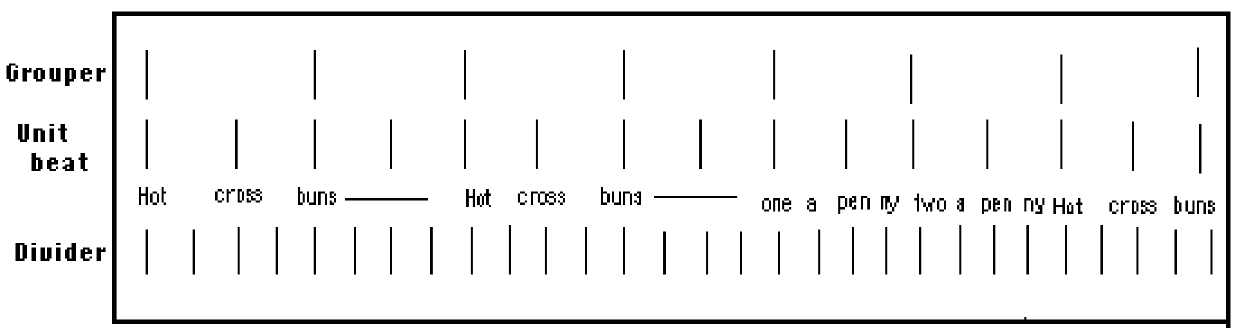

Figure 5. Proportional relations of Hot.

Note that:

[2] Scientists also try to use "natural units," sometimes—such as the atomic mass unit, or the frequency of some basic oscillation, which illustrates the same self-referential strategy. The problem with the scientific use of units is that they often need to measure diverse phenomena. Musical beats, for the most part, need analyze only the "present piece."

- The duration of events on the words, "Hot" and "cross" coincide with, and actually initialize, the unit beat;
- the durations on the word, "buns," are twice as long—lasting two unit beats, thus coinciding with the grouper beat;
- each of the 4 events on one-a-pen-ny, are half as long as the unit beat—they go twice as fast, thus coinciding with the divider beat.

While we do not usually listen just to this underlying temporal metric, it forms the framework within which we hear both coherence and also, as we shall illustrate, the excitement associated with composed perturbations of it.

B. Numeric Representations

Durations are more precisely represented in Impromptu by whole numbers. The general principles are this:

> • Larger numbers represent longer durations, smaller numbers represent shorter durations. The smaller the number, the faster events will follow one another.
> • Proportionality of time can be seen in proportionality of numbers: e.g., durations of 2 following one another go twice as fast as durations of 4; durations of 6 following one another, go twice as slow as durations of 3.

Thus, the beats at the three levels of a typical, duple meter hierarchy can be represented and generated in Impromptu by specifying integers that have a 2:1 relation between each of the adjacent levels of a percussion piece.

Figure 6 shows, as an example, a portion of an Impromptu computer screen where three levels of beats played by three different percussion instruments are producing a typical 2:1 duple meter hierarchy.

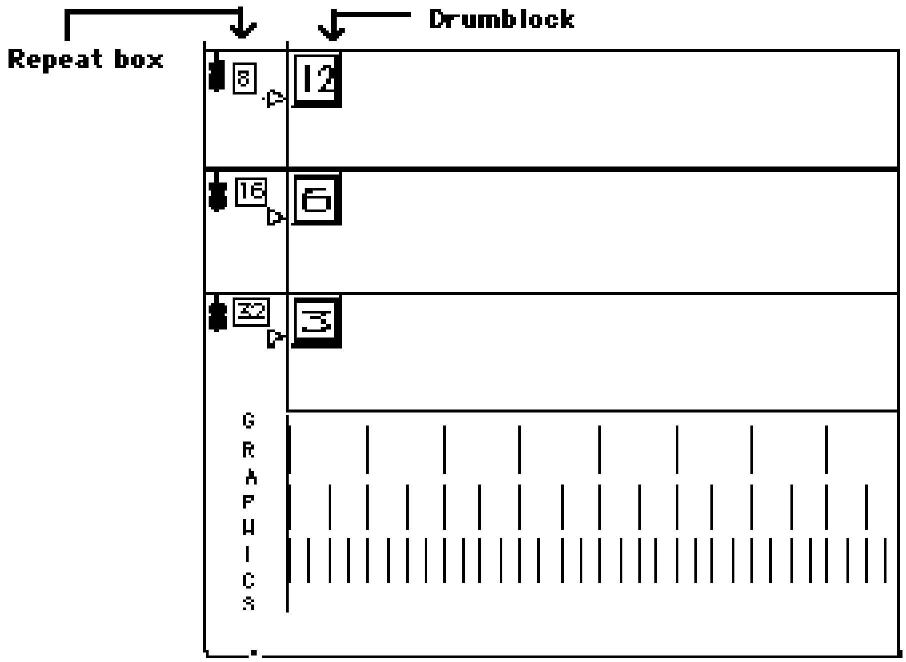

Figure 6. Duple Meter

The large boxed numbers, called "drumblocks," in each channel (voice), specify the duration of events that are repeatedly played by a synthesized percussion instrument—thus generating a steady beat.

The "repeat box" at the left of each voice indicates how many times a drumblock in that voice is to be repeated. As can be seen in the graphics, the 12-block at the top level is repeated 8 times, and it "goes twice as slow" as the 6-block, which is repeated 16 times. The 3-block in the bottom channel "goes twice as fast" as the 6-block, and it is repeated 32 times. The total time in each voice is the same, demonstrating the reciprocal relation between frequency (more properly, repetitions) and period (or duration of the repeated event). That is, assuming a fixed overall length of time, the repetitions specify frequency (number of events per unit time), and the value of drumblocks specifies duration of each event [number of (absolute) units per event], and these are inversely proportional to each other.

In the graphics window at the bottom of the screen, the relative space between lines at each level reflects the relative value of beats in each of the three voices. Thus, since spaces between lines show proportional relations between beats, when the play button in Impromptu is pressed, the sounding events represented by the

[3] In this sense more beats are accomplished.

vertical lines in the middle voice, for instance, will go by twice as fast (twice as frequently) as sounding events in the top voice.

Figure 7 shows an example of just two levels (basic beat and grouper beat) of a triple meter hierarchy. The beat (drumblock) values in this example (6 & 2) have a 3:1 relationship while the repeats (3 & 9) have the reciprocal 1:3 relation. The graphics in this example are an alternative representation (“rhythm roll”) where the time/space between percussion attacks is filled in. Rhythm roll contrasts with the vertical line graphics (“rhythm bars”), where the lines mark just the on-sets (or attacks) of each event. Figure 8 shows the same triple meter hierarchy represented in conventional rhythm notation.

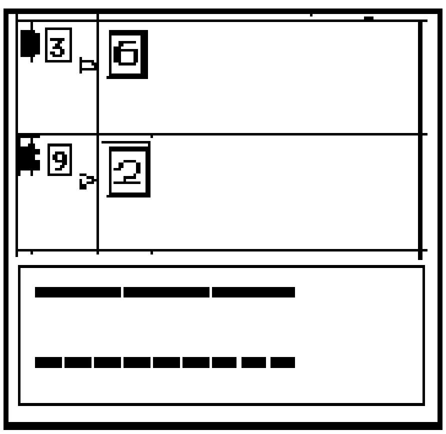

Figure 7. Triple meter
Rhythm bars

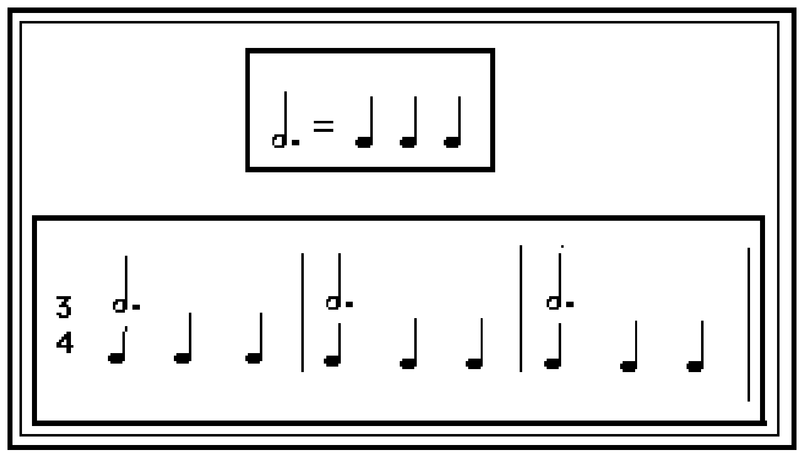

Figure 8. Triple meter
Conventional notation

Four kinds of representations have been discussed thus far—spatial graphics (2 forms), numeric, and conventional rhythm notation. Figure 9 shows the four representations for the same tune, Hot Cross Buns.

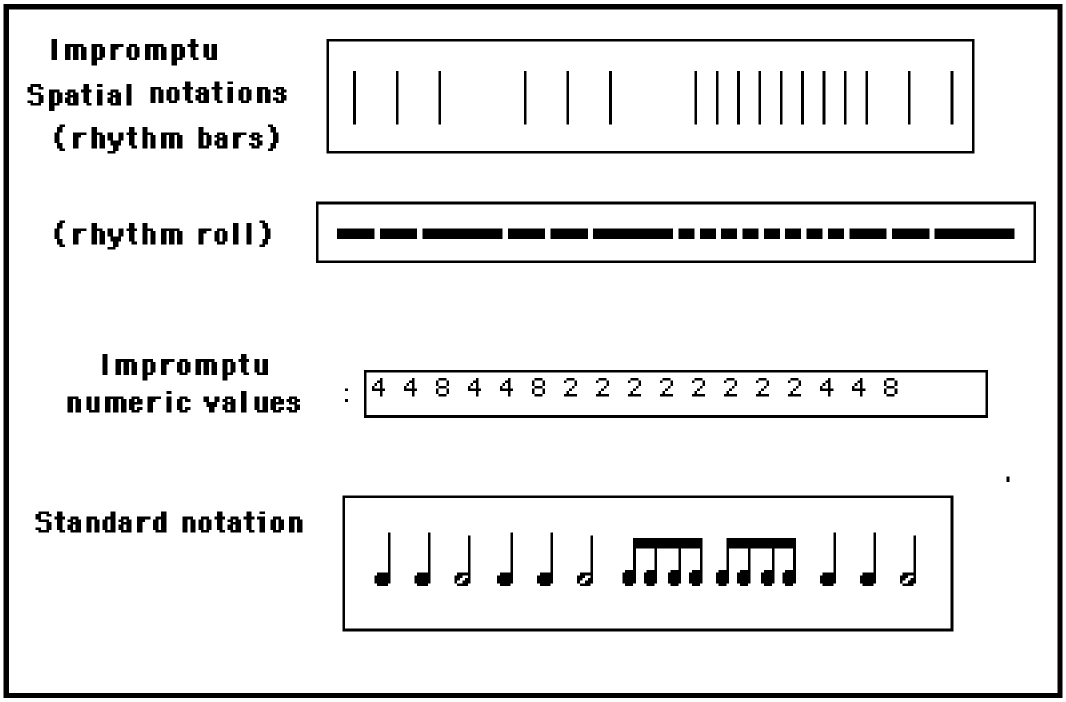

Figure 9. Multiple representations.

Like all representations, each captures some features while ignoring or minimizing others. For example, rhythm bars highlights onsets, which are so critical to listeners' perceptions of music; rhythm roll highlights duration. Note that the duration of the final event is simply not shown in rhythm bars. Standard notation shows metric groups with connecting beams ("one-a-penny," "two-a-penny"), while ignoring others (phrasing, motivic grouping). The numerical representation highlights precise durations and ratios. Reflecting on the ontological differences (the differences among "what they refer to" and emphasize) among these representations brings to the surface the enigmatic nature of representing time and motion as experienced in music, while also pointing to the multiple distinct, but related (and confoundable) aspects of the phenomena, itself. So despite our efforts in designing Impromptu to derive representations in close relation to the common experience of clapping a beat or a familiar rhythm, the elusiveness of representing complex, multi-aspected experience remains. These issues have emerged particularly in our observations of various users of Impromptu. As we will illustrate in what follows, the confusions that arise are often more revealing and enlightening than bothersome.

Multiple representations and the different perspectives they offer are important particularly in an educational (as opposed to professional) environment. Individuals in particular disciplines tend to take the objects and relations named by descriptive, symbolic conventions associated with the discipline as just those that exist in the particular domain. Through practice, symbol-based entities become the objects, features, and relations that tacitly shape the theory and structure of the domain—how users think, what they know, teach to others, and thus what they take to be knowledge. As a result, units of *description* may come perilously close to (pretending to be) units of *perception*—we hear and see (only) what we can say.

The ontological imperialism of homogeneous symbol systems is educationally problematic in at least three ways. First, the discipline is often (or always) much more than what can be easily captured in (small numbers of) conventional representations. For example, novice musicians can "play the notes" but miss phrasing, nuances of emphasis and pace change that distinguish "musical" from "mechanical" performance. Furthermore, the notations do not show novices how to hear even the entities that are most easily depicted. Conversely—and finally—conventional notations may not adequately capture the easiest-to-hear aspects of the phenomenon (Bamberger, 1996).

What, for instance, do we mean by "faster" in musical situations; how would you teach that meaning to one who didn't easily perceive it; and how does it show up in various representations? The language of "going faster (or slower)," in fact, is exceedingly natural and usually spontaneously applied in everyday talk. However, continuing the conceit that it is not obvious, a sensible explanation (word representation) is surprisingly difficult. We might make the presumption that the root meaning of "faster" refers to physical motion—getting to a standard place in less time, or getting to a more distant place in the same time. But, marking a beat literally "goes" nowhere.

We might try to explain that "motion" through time is metaphorically related to motion through space. But this explanation has the fault that "faster" level of the hierarchy of beats doesn't get to the end of the piece any more quickly.[4] The most obvious description of "faster" here is "more beats per unit time." But this presumes the

[4] Although this is not a paper about word semantics, we feel it is plausible that the root meaning of "faster" that makes it transparently descriptive of both musical pace and physical motion is that "more is happening in a given time." In the case of motion of objects, more distance is accomplished, and in the metaphorical sense more beats are accomplished. A more concrete explanation of the connection between the senses of faster is that in our common experience of running, increased frequency of steps (beats) is associated with increased speed of locomotion. So it is easy to "read" increased frequency (beats per unit time) as "increased speed."

understanding of a technical concept, frequency, and in any case does not account for the intuitive obviousness of the characterization that we would like to achieve.

Let us turn to graphical representation of “faster”: In Impromptu, faster is shown numerically by the numeral on a drumblock, the beat value, which shows the “duration” of the events that are repeatedly played. That convention is both highly functional and useful in that it leads directly to important mathematical insights about music (see later sections). But, it might well be viewed as “unnatural” by scientists, who see “faster” better expressed by frequency (events per unit time), which varies inversely with duration. Doubling a beat value halves the frequency (“per unit time”) of beat events. Indeed, many people, not just scientists expect a bigger drumblock number to correspond to a faster pace.

Ontologies are difficult things. Descriptions or representations are, at best partial. Certainly some make better starting points, perhaps connecting better to naïve experience (e.g., rhythm bars and clapping). Some, also, make better conduits to normative representations and expression (numbers). Others, and relations among representations, might raise good questions, initiate good inquires. We don’t systematize or settle these issues in this paper, but highlight them and the deep cognitive issues they represent in the data to come.

One additional brief example will show a more realistic problem of representation, ontology, and instruction. Consider an instance of musical terminology—a conventional definition of triple meter.

> "3/4 meter (or 3/4 time) means that the basic values are quarter-notes and these recur in groups of three. Such metric groupings are indicated by bar-lines that mark off measures." (Harvard Concise Dictionary of Music, 1978)

Notice that the definition is intra-symbolic, exclusively in terms of the symbols of conventional notations, themselves (“3/4,” “quarter notes,” “bar-lines,” “measures”). The definition is, so to speak, about the notation more than about the musical phenomena being represented. Such definitions finesse the fundamental issue of how one perceives the given relations in favor of how one denotes them. “Limiting vision” to formally notated aspects is particularly problematic in music since, in the service of giving concise performance instructions, the notation leaves out critical aspects of the coherence directly experienced by the listener.

Definitions in terms of representations hide ontological aspects of the experienced phenomena. One cannot literally hear quarter notes or bar lines, so the perceptual objects to which these symbolic objects refer are not even obliquely referenced in the definition. By the same token, conventional music notation makes it more difficult to go beyond the features represented by these conventions to other phenomena that share similar underlying structures (harmonics in sound analysis, gears, pendulums, patterns in laying multiply-sized tiles).

We conjecture that definitions that finesse perceived relations in favor of how one denotes them severely limit the sense students can make of mathematics and science. In this regard, diSessa and Sherin (1998) have argued that the essence of understanding some scientific concepts lies precisely in developing strategies that allow for the perception of ("noticing") the entities and relations of the conceptual fields in question within the varied phenomena in which those entities and relations occur. We believe this is a deep, rather than accidental relation between music and science, and that the cognitive theory of music shares much with that of mathematics with respect to representation and ontology.

WORKING WITH CHILDREN

Organizing Time

The examples that follow illustrate how the group of six 6th grade children with whom we worked were sometimes guided by Impromptu's multiple representations and appeal to multiple sensory modalities. In addition, confronting differences in representation stimulated provocative questions as the children interrogated one another's work. As they developed projects in this environment the children discovered principles of embodied mathematics in the common music all around us and also went on to use their discoveries to create original melodies and rhythms. The initial examples focus on rhythm where the mathematics is most clear. More subtle and perhaps more interesting intersections between music and mathematics were discovered as the children composed melodies—particularly as the graphic representations helped them come to consider patterns such as symmetry, balance, grouping structures, orderly transformations, and *structural functions*. Structural functions include, for instance, pitch/time relations that function to "create boundaries," or entities (e.g., phrases) some of which sound "incomplete" and thus function to move a melody onward, in contrast to entities that sound "complete," thus functioning to resolve or settle onward motion. Structural functions are not directly shown in either conventional notation or in Impromptu's notations. And yet, as we will show, these structural differences are immediately noticed by children who have grown up listening to the familiar music of this culture.

Examples

Example 1a: By the third session of the project the children were generally familiar with Impromptu's proportional rhythm notation and with the computer synthesizer's percussion instruments. This session began with the children, as a group, playing real drums. One child played a slow, steady beat on a large Native American drum. We asked the others, using claves, woodblocks or just clapping, to play a steady beat that went "twice as fast. With just a little guidance, the children were quite quickly able to create the two levels of beats.

Then we asked the children to use the computers to make a drum piece such that two of Impromptu's percussion instruments, each playing its own part, would play beats that were related to one another like the beats they had just played on real drums. That is, they should experiment with Impromptu drumblocks and pairs of percussion instruments so that one of the instruments in one channel would be playing "twice as fast" as the percussion instrument in the other channel. They should find as many different pairs of drumblocks with this relationship as they could.

Figure 10 shows examples of Sam's and Anna's first solutions for the task.

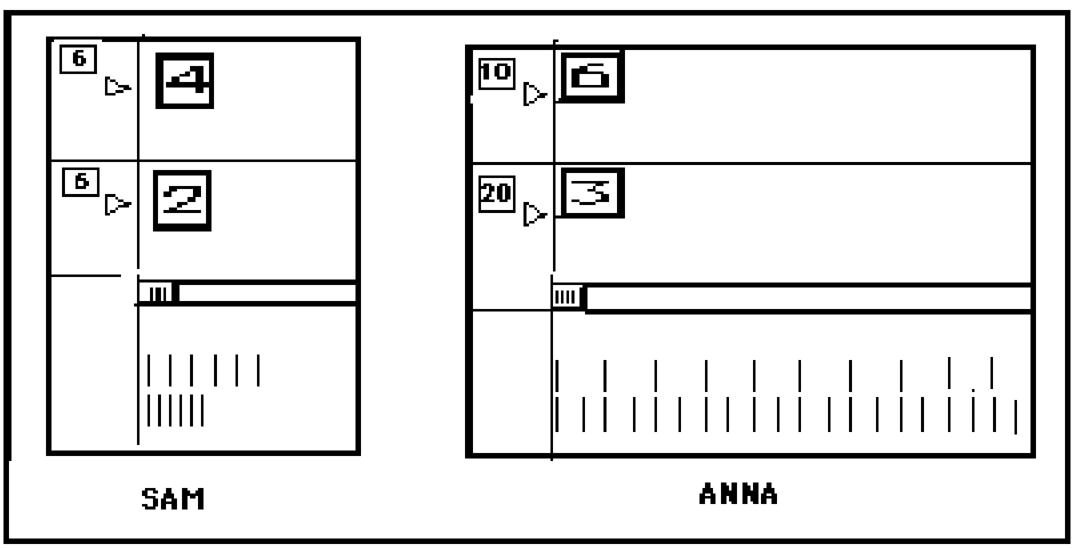

Figure 10. Sam's and Anna's "twice as fast."

Sam has 6 repeats for his 4-beat and also 6 repeats for his 2-beat. Anna makes 10 repeats for her 6-beat and 20 repeats for her 3-beat. Max, listening to the two examples, had an interesting question: "How do you make them [the instruments] come out even, 'cause Anna's do, but Sam's faster beat stops too soon?" Anna explained that, "...like 3 is twice as fast as 6, so the 'repeat' has to be twice as much, too." Sam tried it—making his 2's repeat twice as many times (12) as his 4's (6). He also switched to the rhythm roll graphics in order to see more clearly that the two drums really did come out even (Figure 11).

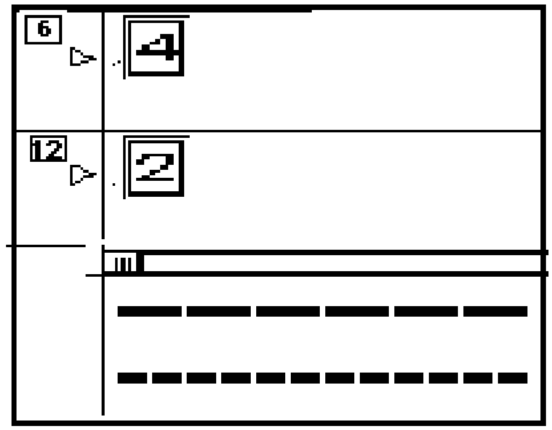

Figure 11. Sam's new solution.

Listening to the result, Sam had a different way of explaining what he heard: "It works because the 2 is *half as big*, so it gets *twice as many* repeats as the 4. I mean, the 'twice as much' is the same but it's in reverse—4 is to 2 like 6 is to 12 only upside down." We think it is quite likely that the evident spatial relations of size and number in the rhythm roll graphics supported Sam's insight and way of talking about it. One can literally see (if one is attuned to such things!) "half as big" and "twice as many." "Upside down," on the other hand, refers to vertical placement of the duration numbers (drumblocks) on the Impropmtu diplay in relation to the repeat numbers (and/or possibly the spatial representation of standard fraction numerals—note Sam's "formal" language: "4 is to 2 like 5 is to 12"). The design of Improptu, with corresponding numbers in a vertical relationship, encourages making the connection to standard mathematical presentations of ratios or reciprocal relationships.

Sam, learning from Anna, had discovered that there is a reciprocal relation between duration of events (how *much*) and number of repeats (how *many*). That is, if the total time is the same for both instruments (they "come out even"), the ratio for the durations and the ratio for the number of repeats is the same but, as Sam said, "in reverse" (or "upside down").

These students are exploring and describing relations among particular kinds of phenomena in a specific situation. In order to confront the implicit ontologies of representations and perception, let us couch the children's descriptions in the mathematical terms of the following equations:

$$mx = c$$
$$ny = c$$
$$(x > y)$$

We emphasize that the equations do not express the same meanings or even (necessarily or transparently) refer to the same kinds of entities as the children's descriptions. Specifically, The power of the equations lies precisely in generalizing beyond any particular phenomena and any specific situations. One could go on to specialize the meanings of those equations with the following statement: This pair of equations expresses the accumulation of a number (m and n, respectively) of instances (measured by x and y, respectively) that total a constant amount (c). While This description is still more abstract and general than the students' meanings. However, it is more specific to this case than the pure algebra because the terms are intended to refer to things to which the students are responding, such as iterations, accumulation, and total. But the equations miss what Anna sees directly something that is only implicit in the equations: She says "…3 is twice as fast as 6, so the 'repeat' has to be twice as much, too." To relate this to the equations, we must do several things: First, we must connect the second equation, $ny = c$, with the size of y (smaller) and the intuitive perception that this condition is "faster." Then, we must connect this with the inference that the faster occurrence (involving y, which is less than x) must receive more repetitions. Thus, in equations, Anna is stating that n *must be greater* than m). In fact, "faster by a factor of two" translates into "more by a factor of two."

That Anna has an intuitive feel for multiplicative, as opposed to additive, relations seems fair enough and this is not a trivial accomplishment in itself. Anna is almost certainly not learning that here, although other students might be (in particular, Sam might be, though he did express his own version of the sense he made of this constraint after it was pointed out to him). However, neither Anna nor Sam's explanations have any of the generality of the formal mathematical statements. Both Anna and Sam's descriptions apply specifically to this situation and what they are hearing as influenced, in part, by the particular notations of Impromptu: Numbers afford easy expression of multiplicative relations, and also, for example, Sam plausibly saw "half as big, but twice as many" in rhythm roll graphics. Put most directly, Anna, with Sam learning from Anna, have succeeded in recognizing in this situation, relations that are reminiscent ("correctly" and insightfully so) of relationships they have seen before (and will see again) in their "school math." The significant thing is that they have, indeed, recognized the relations while working in an entirely new medium, and they have been able to put them to work in this new situation; the particular, musically important work to be done is "to make things come out even." This is, we believe, a move in the direction of generalizing.[5]

However, we can be quite sure that the students cannot generalize these relations in the way expressed in the formal mathematics, and this raises interesting and fundamental questions: How can we describe the specific ontological differences between the students' understanding and the implications of the mathematical formalisms; what is the nature of the transformations involved in moving from limited practical situations to generalizations such as expressed in the proposed equations? And why is this move apparently so difficult?

Backing off from intractabilities, what we do see (we believe irrefutably) in the students' work is the following: Some students (Max and, initially, Sam) do not immediately perceive the relations that Anna notices immediately upon being questioned by Max. But in the context of sounding events coupled with the use of graphical and numeric representations, they are able to generate, perceive, and thus validate these relations. From this we infer that (1) the multiple Impromptu representations and their immediate sound-back in familiar musical structures can help students understand (possibly, generalize) the basic relations involved, and (2) these can be steps toward understanding proportional reasoning robustly in a range of situations.

Example 1b. Joe made several pairs of drum beats that worked: 10 and 5, 8 and 4, 6 and 3 16 and 8. Playing back what he had made for the other children, Joe said, as if just telling the obvious, "Well, they're just equivalent fractions"! Joe again made a direct connection between sounding rhythmic structures and school math: the equivalence of equivalent

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[5] Notice that generalizing amounts to adding a new, particular way of interpreting some basic mathematical relations of proportionality. This view of "generalizing" (bringing to more contexts) *not* by abstracting, but by adding "concrete" instances of reasoning is inherent in the ideas in diSessa & Sherin, 1998, and also in some more recent work on "transfer" (Wagner, 2003).

fractions could be heard! We don't take it to be a trivial matter that this child has found a context in which the equivalence of fractions is directly salient and also powerful (identifyin things that "sound the same")—in contrast to an inference based on rules that have been memorized. See also, Arnon, Nesher, Nirenburg (2001).

Overall, the children, working in an environment using joined media (numbers, spatia representations, and sound-in-time) were able actually to generate coherent structures using their understanding of the principles of ratio and proportion expressed and experienced in novel situations; they were learning about the reciprocal relationship between how much (duration) and how many (frequency); and they were learning the connection between equivalent fractions and proportion embodied by pairs of iteratively sounding events that ar different in absolute "speed," but the same in their internal relations.

We played out these relations with other students in a different modality. First, we marked off 8 equidistant lines (about 1 foot apart) on the floor. Then, while one child played a slow beat on a big Native American drum, two other children, side-by-side, walked "in time" with the drummer. The children were given the following instructions: "One of you (Rachel) will walk along stepping on each line in time with the drummer. You (Simon) will take two steps for each one step of Rachel's and two steps for each one of the drummer's beats. *But* you have to arrive together at the end of the lines." Thus, the students were doing in action—literally embodying—what Anna and Sam did using the computer and the trace left by the rhythm bar graphics. When the children had reached the end of the marked lines, we asked, "So who was going faster, Rachel or Simon?" The first and immediate response from several children was, "Simon!" But then Simon quickly added, "But we arrived at the end together." Considerable discussion followed. Agreement was finally reached when Steven proposed that "Simon's feet were going faster, but their heads were going together."

Of particular importance is the effectiveness of the activity and the environment (whic includes, again, sound, action, and periodicities as units of measure) in externalizing what might otherwise be tacit dilemmas. Specifically, the students are working to stabilize the multiple possible senses of going faster—"attend to feet," or "attend to heads." Although we do not pursue this stabilizing here, later on, playing with huge cardboard gears and also with pendulums, these same children were able to distinguish linear from rotational speed, and between linear speed and frequency: Linear speed—the number of teeth passing a point in unit time—is preserved in contact between a bigger and smaller gear; but rotational speed—revolutions per unit time—is not. A pendulum offers a similar challenge. As it winds down, linear speed decreases, yet "speed of repetition" stays the same. (For more on the children's work with gears and pendulums see Bamberger, 1990 & 1998.)

Example 2: During the next session in working with Impromptu, we (JB) suggested the children try a beat with a duration value of 4 (a 4-beat) in one percussion instrument and a 6 beat in another. See Figure 12. Listening to what they had made, they agreed that it sounded "really cool."

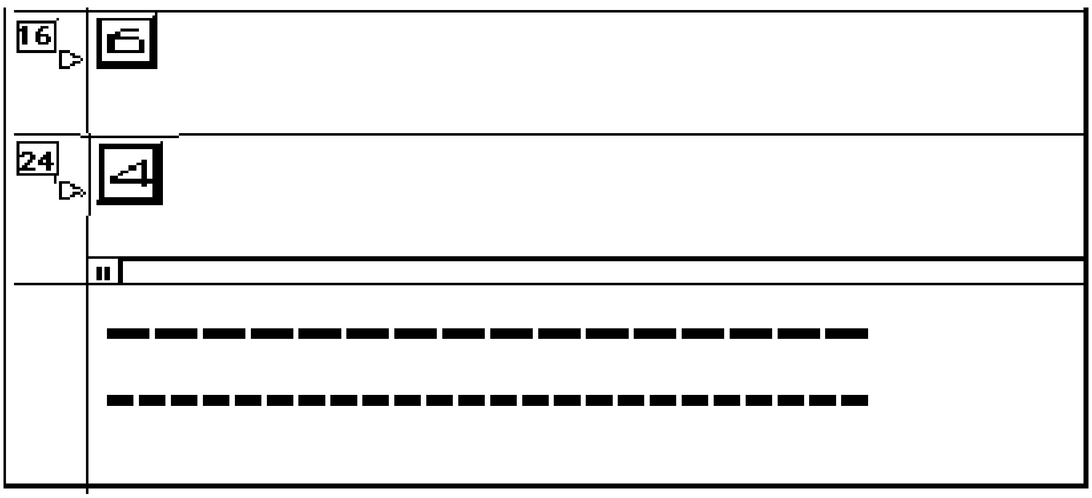

Figure 12. 6:4 sounds "really cool."

Going on, we asked, "So where do the two drums meet? Where do the 6-beat and the 4-beat come together at the same time?" Using the rhythm bar graphics to make it easier to see where events came together, Kathy said, "They meet at 12" (Figure 13).

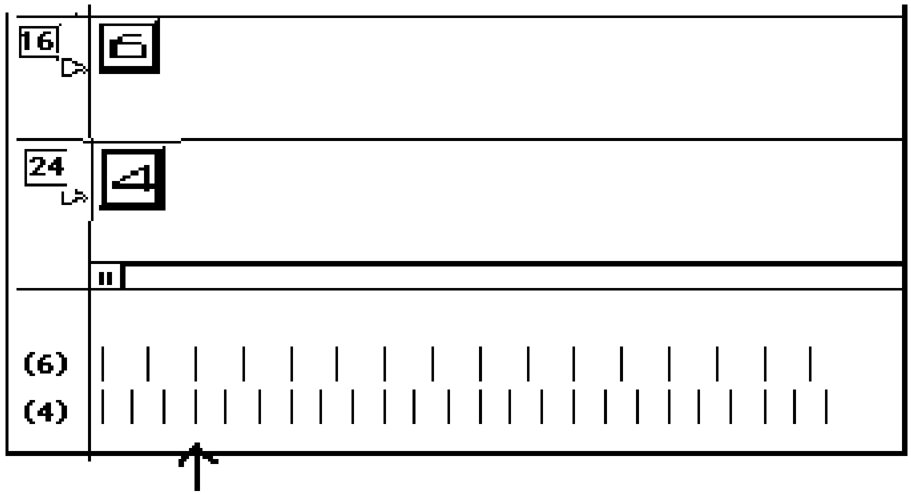

Figure 13. "They meet at 12."

When Kathy was asked how she knew, Joe interrupted to say, "Oh that's that least common multiple stuff!" To test if we could really hear this "least common multiple," we added a third instrument playing the 12-beat (Figure 14). Listening to the result, it was as if the 12-beat "pulled the other two beats together." Once again, perceived rhythm met school math.

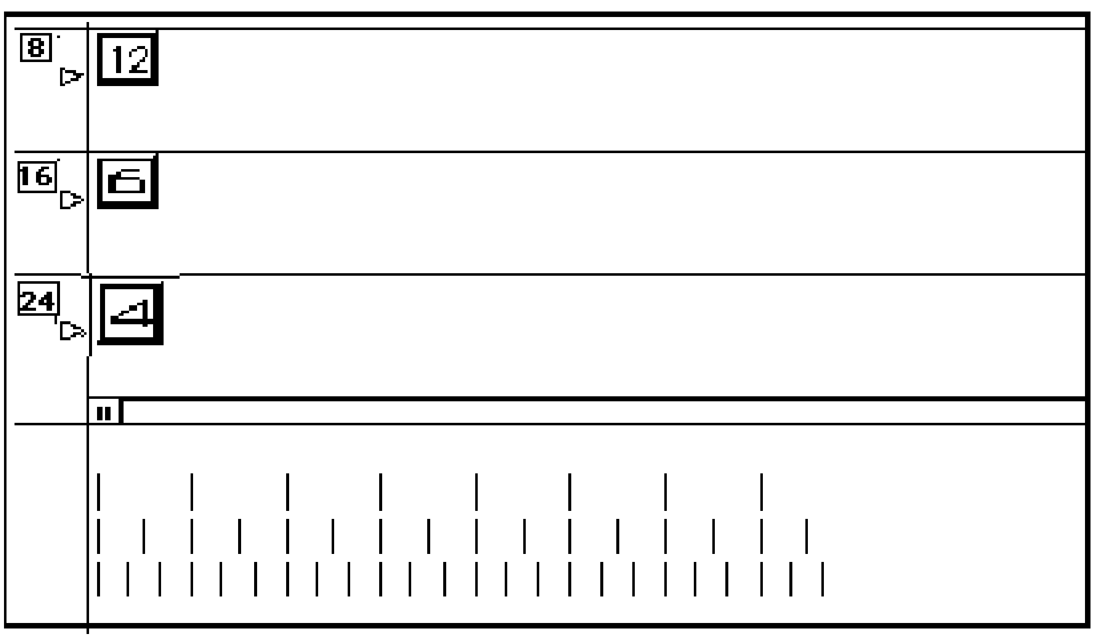

Figure 14. "Least common multiple stuff."

It is important to note that the coincidence of periodicities is not intrinsically about leas common multiple. But when experienced in a context that numericized the control of repetitions, coincidence came to be about least common multiple.

Further discussion and experimentation revealed more connections between mathematics and music: For instance, there were two 6-beats for every three 4-beats. "Well, of course, because it takes 2 6's to make 12 and 3 4's to make 12: 6 x 2 is 12, and 4 x 3 is 12." Looking and listening, those expressions became more clear as they became sound in action: You could see and hear that 6 x 2 means, "*d o* 6's, 2 times," and 4 x 3 means, "*do* 4's, 3 times." We could also see and hear, once again, that the bigger number and the slower beat needed fewer elements (2 of them), while the smaller number and the faster beat needed more elements (3 of them). Moreover, the 2:3 in number of beats per common multiple was the same ratio as the value of the beats, 6:4, "but upside down." And finally, the number of repeats in each instrument, 16:24 was the same ratio as 6:4 but still "upside down." And all for the same reasons: bigger/slower events need proportionally fewer elements than smaller/faster events to be equal in total time.

So why did the drums sound so "cool"? This is an example of rhythmic tension, "excitement" as described earlier. In this case, there is a tension or conflict in a mismatch. The second of the duple meter (6) beats "misses" the background triple meter (4) beat—whic gets resolved in a convergence at regular time intervals (on the 12-beat). We might say that, on the way to the common slowest beat (or the common multiple), there was tension (2 against 3); yet, that tension is neither confusing nor chaotic because it is always quickly resolved. In all, the rhythm was more interesting/exciting than the regular alternation (as ir 2:1). Stravinsky uses exactly this metric conflict with its regular resolution at the common multiple ("dotted half note") to striking effect in Petrushka, where he pits a triple meter Viennese waltz tune against a compound duple meter accompaniment (for more, see Bamberger, 2000).

Listening carefully to the computer version along with the graphics, the children managed to play the 2:3 rhythm on their percussion instruments. Reflecting more generally about representational affordances, we note that the mathematical/musical inquiry into the relationships we heard would hardly have emerged if we had been using the conventional representation of compound duple meter against triple meter as shown in Figure 15.

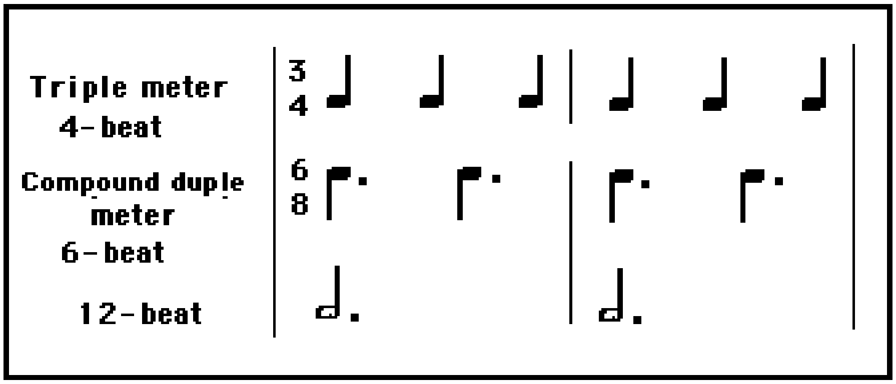

Figure 15. Metric conflict: Triple against duple.

The usefulness of a representation, of course, depends on the purpose for which it is intended. For example, on one hand, if the meanings of conventional notation symbols hav been internalized—say we are dealing with conventionally conversant performers—playing the rhythm from the notation in Figure 15 would be much easier than interpreting the Impromptu numbers, 6:4, especially without the graphics.

Notice that in Figure 15, for example, the unit beat in duple meter (or more technically, compound duple meter)[6], is notated as a "dotted quarter note," while in triple meter, the unit beat is notated in a different way, as a "quarter note." The two different unit-beats share a common slower beat, the common multiple notated as a "dotted half note." However, the relation "common multiple" is obscured in the notation in part because the representation i limited to conventional note symbols rather than their implicit arithmetic relations. Of course, specific note names are internalized and effectively used by professionals. However, that efficiency comes at the cost of clarity with respect to more general mathematical structure. Moreover, in playing from a score, a professional scanning a passage such as this, uses familiarity with the specific, local spatial pattern of the conventional notation (rather than "a note at a time") not at all the calculations or the potential generalizations that are

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[6] The term, "compound duple meter" refers to the triple grouping of the unit beat. As indicated by the 6/8 meter, the underlying unit is an 8th note. Howeved, the grouper beat must then be notated as dotted quarter note, which includes three 8th notes. The two dotted quarter notes in a "measure" convention accounts for the term, "compound." Understandably, this is very confusing for beginning students, and almost entirely for representational reasons.

implied. Conversely, it is exactly Impromptu's proportional, integer notation that led to the children's insights concerning common multiples and proportional relations.

The children went on to use what they had discovered in these experiments as the basis for composing percussion accompaniments for melodies. The projects involved first listening to a melody played by an Impromptu synthesized instrument (flute, clarinet, vibes, etc.), then finding proportional values for beats at three levels of a metric hierarchy that fit with the melody. Using the found hierarchy as a framework, they children composed patterns of varied durations played on percussion instruments that reinforced the hierarchy, as well as accompaniments that created conflict (but not chaos) with the rhythm of the melody. They agreed that making just the proportional relations "sounded good" but was boring.

Composing Melodies: Embodied Patterns

At the beginning of a later session we introduced an idea that is powerfully shared by structures in both mathematics and music—looking and listening for patterns. We began with the question, "What is a pattern?" Sam answered, "Something that's repeated more than once." After a moment, Katherine said, "But 1, 3, 5, is a pattern because it skips one every time." We left the meaning of pattern hanging for the moment, but intended to come back to it. Their previous insights—common multiples, equivalent fractions, reciprocal relations, proportion, ratio—are also patterns, of course, and like most patterns, these involve noticing relationships that maintain their integrity across media and sensory modalities.

Focusing, now, on melodic patterns in preparation for composing melodies, we asked the children to listen to some short melodic fragments—called "tuneblocks" in Impromptu. We begin melodic composition with these short but structurally meaingful elements because research has demonstrated that, in contrast to conventional music notation where the *units of description* are individual "notes," intuitive *units of perception* are at the more aggregated level of whole melodic fragments (Bamberger. 1991; 1996). Indeed, "tuneblocks" represent the same level of musical structure as the very early *neumes.* Figure 16 shows an abbreviated version of the Impromptu Tuneblocks screen for composing with the set of tuneblocks called "ELI."

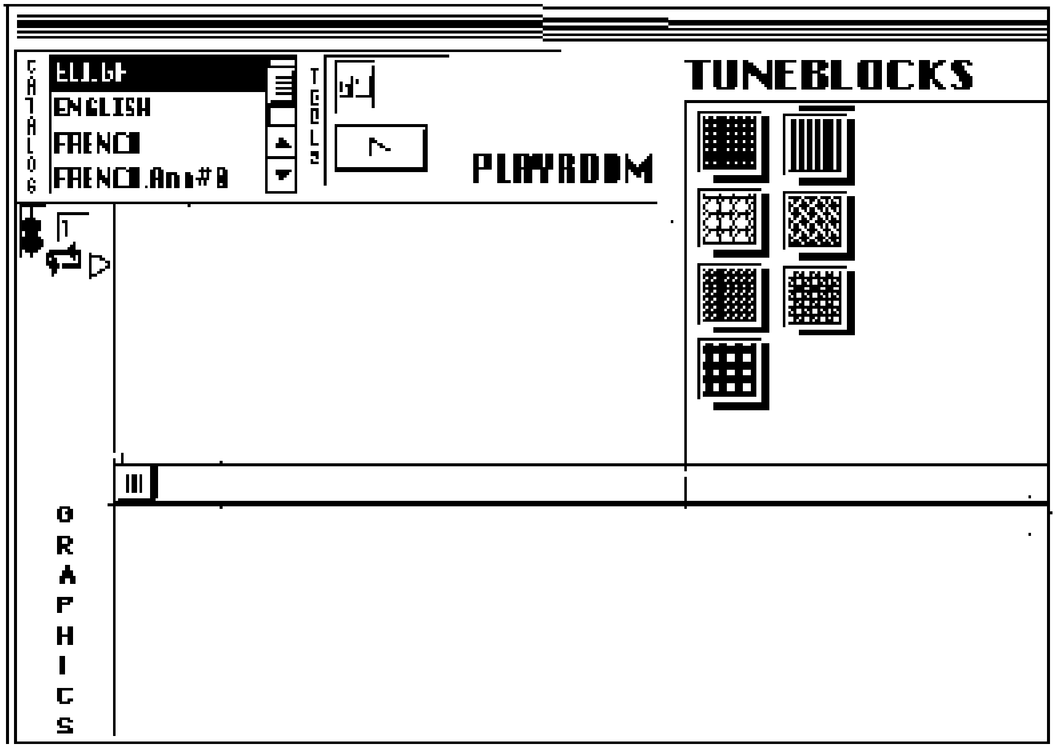

Figure 16. Impromptu Tuneblocks screen.

Tuneblocks can be heard individually by clicking the icons in the Tuneblocks area. Th designs on the icons are neutral graphics with no reference to the melodic "shapes" that the blocks actually play. The intention is to focus students' attention on their own musical perception, *listening* to the melodies rather than looking at partial representations. To build tunes, blocks are dragged into the "Playroom," arranged and then played back in any chosen order. Blocks placed in the Playroom can be seen in several kinds of representations in the graphics area.

To make it easier to refer to the blocks, we gave them number names from 1 to 7 according to the order they happen to appear in the Tuneblocks area as shown in Table 1

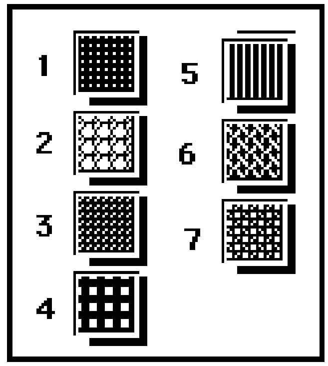

Table 1. ELI Tuneblocks.

Asking the children to listen for patterns, now, we clicked Block 1 and then Block 6 in the Tuneblocks area. (Remember, the children were *only listening* with no visual cues yet—not even Impromptu graphics.) Anna said, on hearing Blocks 1 and 6, "The rhythm is the same," but several other children immediately insisted, "No it isn't!" Exploring the source and meaning of this disagreement would be a continuing concern, but in the short term, we suggested that the children could experiment by clapping just the rhythm of each o the tuneblocks. Listening to their own clapping, the children agreed that the rhythm of the two blocks sounded "pretty much the same." To test further, we listened to the two tuneblocks again, this time dragging them into the Playroom area so we could listen and loo at Impromptu's rhythm roll graphics while the blocks were playing (Figure 17).

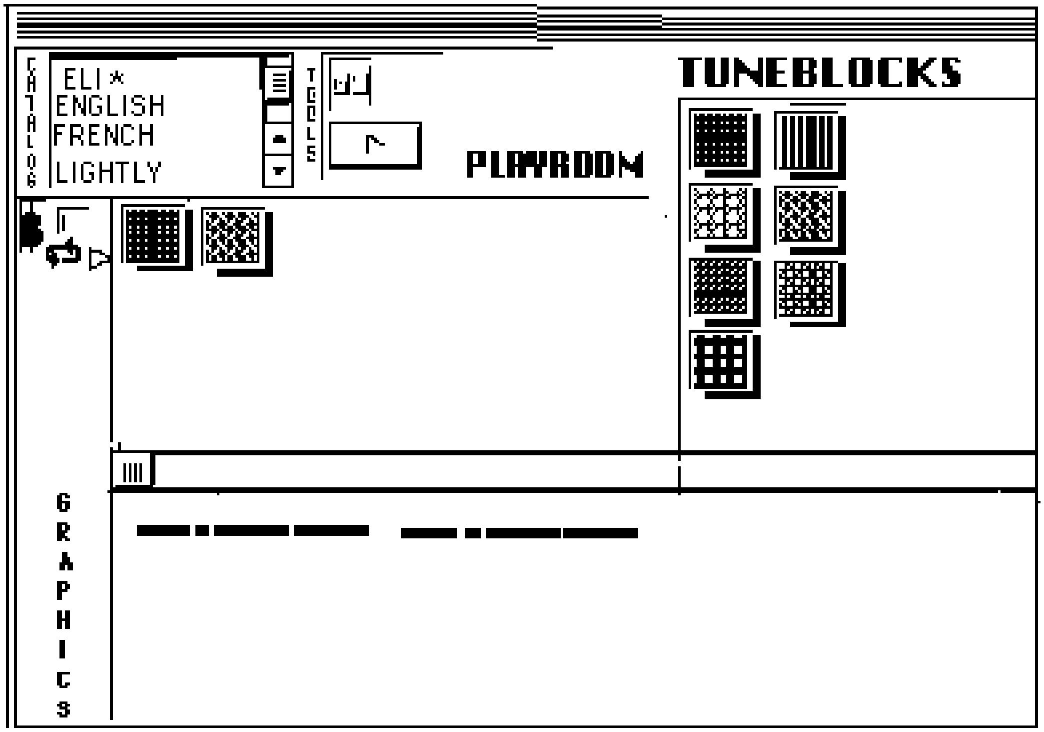

Figure 17. Blocks 1 and 6 in the Playroom with rhythm roll graphics.

Joe agreed that the rhythm of the two blocks *looked* exactly the same, but then he asked, almost petulantly, "Well then, how come they sound so different?"

Looking this time at a different representation of the same blocks—pitch contour graphics (Figure 18), the children noticed differences: Block 1 "just goes down," but Block 6 "goes down and then up," and both blocks "end in the same place."

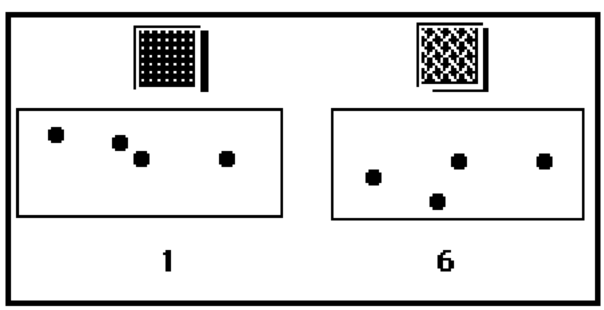

Figure 18. Blocks 1 and 6.

While not arriving at a complete answer to Joe's question, (which continues to tease music theorists, as well—e.g., Hasty, 1997), just working with these two blocks and looking a different representations, the children were able to shift their focus between two dimensions of the same melodic fragment—pitch and duration (rhythm). After some discussion the children did conclude that it had to be the differences in pitch (the "ups and downs") betwee the two blocks that made the same rhythm sound different. Moreover, in terms of level of detail, in first listening to and looking at the block representation, their focus of attention ha been on the integrity of the entities as a whole. As they looked for patterns and compared pitch contour and rhythm roll graphic representations, their focus moved down the structural ladder (from the block level) to greater detail—to duration and pitch (the note level). Differences in representation, and their own actions (e.g., clapping "just the rhythm") disaggregated the two properties, duration and pitch, which before were simply absorbed into the gestalt of the structurally more aggregated tuneblocks.

In more general theoretical terms, we believe it is appropriate to say that representation and operationalization processes (e.g., representing-in-action) psychologically *create* the separate aspects; they don't just "reveal" or demonstrate them. Working in the Impromptu environment, graphics along with other tools helps in this process of disaggregation and with it the emergence of new aspects by making it easy both to see and to hear, as in this instance, just the rhythm or just the pitch of a block. Perceptual influence across dimensions without these facilities makes such a process much more difficult than might be imagined. I fact, while listening to the unfolding of a melody, it is exactly this confluence, the perceptual inseparability of dimensions, that gives an event in the moment its particular "meaning" or function.

In technical terms, we would describe this as a perceptual influence across parameters or across aspects. That is, patterns heard in one parameter (e.g., pitch) influence or disguise patterns perceived in another parameter (e.g., rhythm) as compared to when one or the othe aspect is somehow isolated so as to become the single focus of attention. For some specific examples, see Bamberger, 1996. This basic phenomenon undoubtedly reflects, at least in som instances, why experts cleanly see "the structure" of two instances of some phenomenon as identical, and yet novices do not. Identity, similarity, and the disguising effect of context (and how to sort these out) will continue to play a role in later discussion about tunes and the relation of fragments composing them.

Functions, Fragments, and Transformations: What Makes an Ending?

Once pitch and duration were differentiated, the children had a basis for noticing new, rather subtle patterns of similarity and difference in other pairs of blocks. For instance, listening to Blocks **2** and **5** while watching the pitch contour graphics (Figure 19), Max, who was a very shy child, quietly said of this pair, "The second one [**5**] sounds ended but the first one [**2**] doesn't."

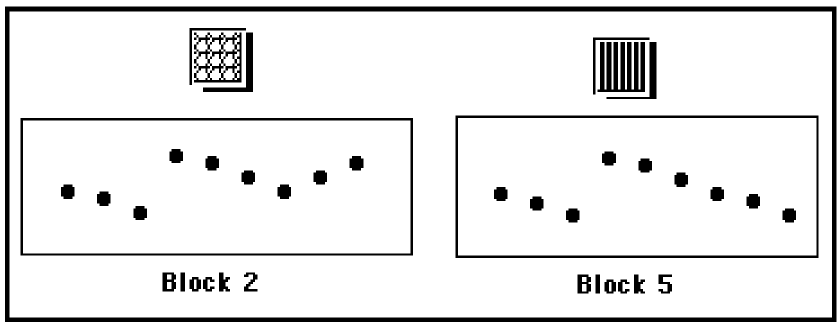

Figure 19. Blocks 2 and 5.

Playing the pair again, the children agreed with Max, but then Kathy made a surprising discovery: "But all the notes are the same in both of them except for just the last two!" This prompted the same question as before: "Well, then, how come they sound so different?" But added to that question was, "And what makes something sound ended, anyhow?"

As the children listened to these two blocks, comparing them with a focus on patterns, their attention had shifted to differences in *structural functions* (e.g., tension, moving onward, in contrast with resolution, arrival), along with a very basic and critical question: what makes a certain pitch sound stable, resolved, "ended"? Once again the children had encountered a situation that raised questions central to our perception of musical structure. We did not pursue this path very far with these children. From our experience with college students confronting the same questions we have learned that it takes a lot of inquiry and experimenting before they can arrive at even a tentative answer. Indeed, while music theorists give names to this focal pitch (a "tonal center"), the question of why, in much Western music, only one pitch in a given pitch context is heard as generating an ending, is one to which theorists continue to seek more consistent and causal answers (e.g., Dahlhaus, 1990). And yet, to hear an ending or resolution function in the familiar music of our culture is something even very young children can do.

One way we have tried to explicate this seemingly intuitive but culturally-specific, learned phenomenon, is with the following experiment: Impromptu makes it easy to use an entirely different set of pitches to play a sequence of notes that keeps the *internal* pitch and time relations the same as in the original tune. Now the tune maintains its identity (it is heard as the same tune), but listeners hear a different pitch as the most stable—i.e., a new pitch has acquired the "ending" function, "tonal center." In music theoretic terms, changing a pitch collection but keeping internal relations (pitch and time intervals) the same is called *transposing* the melody. It was easy to program the Impromptu software to transpose, exactly because transposing is an example of a rule-driven transformation.

Going on with the children's focus on patterns, we listened to Block 4 from the ELI set. Surprisingly, Kathy noticed that Block 4 was, "...a piece of block 2—the end piece—with the rhythm changed" (Figure 20).

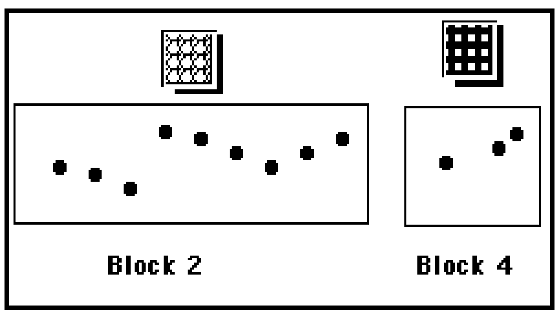

Figure 20. "…a piece of block 2."

Music theory refers to this kind of modification as fragmentation. Fragmentation is one of a group of transformation techniques whereby composers preserve some aspect and thus a sense of cohesiveness, while generating (and allowing further) variation, often generating a new structural function. In the case of fragmentation, the fragmenting of a melodic entity also increases the rate of events—that is, boundaries of entities occur more quickly as we shall see later in Kathy's composition.

The focus on patterns had led to hearing both similarities and differences in comparing blocks. Patterns did include repetition, but also patterns of change—like Kathy's 1, 3, 5 pattern. For instance, listening to Blocks **3** and **7**, the children said that Block **7** was just Block **3** "shoved down" (Figure 21).

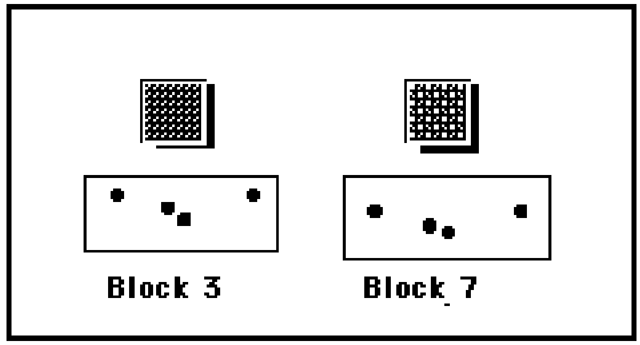

Figure 21. "…shoved down."

This is another kind of very common transformation of a given entity where the pattern of pitch and time relations remains intact, but the whole pattern starts one step lower (or higher) along the scale—it is literally "shoved down" in conventional or pitch contour representations. Once again, the Impromptu pitch contour graphics helps to make this relationship quite vivid for students. Called a "sequential relationship," Baroque composers, particularly Vivaldi, often used it as the basis for extending whole compositions. (Sequential relationships are not necessarily transpositions, since pitch intervals are not preserved in "moving a note sequence" up or down the notational staff.)

Abstracting a core mathematical structure

This section is different than the previous and next in that it seeks to draw out some mathematics that the students (and possibly some readers) do not see in the music. It is provocative (we hope) in setting a future agenda of further exploring what mathematics is implicit and might be learned in the context of music, and in terms of what mathematics might be productively used in thinking about music. It is clearly speculative in that we have not tried to "draw out" this mathematics, and we do not know exactly what aspects of a computationally supported context might facilitate it, in the way Impromptu notations seemed to support student appreciation of the inverse proportional relationship of "how much" and "how many." This provides an "experiment in waiting." If we can succeed in drawing out and making this mathematics functional, will that work in the same way as the above (and below) instances; if we can't, what is different about this mathematics?

The mathematical structure at issue underlies two of the central phenomena encountered above. In particular, it underlies the easy and natural ability of children (and adults) to perceive rhythmic invariance under a change in the *tempo*—that is, the rate of the underlying beat (or the duration of the temporal unit). For example, the children had little difficulty producing multiple examples where the proportional relations between beats (2:1) stayed the same but the absolute durations of beats, and thus the tempo, varied (e.g., 6:3 or 10:5). Similarly, the children had no difficulty hearing invariance when pitch contour is maintained but shifted along the pitches of a given scale, as in the commonalties between the two blocks depicted in Figure 21. More dramatically, it accounts for transposition—the perceived invariance when a tune is played "in a different key." Even though one uses a different set of absolute pitches, as long as the internal relations of pitch and time are kept invariant, listeners hear both versions as the same tune. Indeed, if the two hearings are sufficiently separated in time, listeners may not notice the difference at all!

A simple model of the mathematical structure we seek to explicate is to imagine a "thing" that contains "pieces" and "relations among pieces." For example, the thing might be a melody, or it might be a drum piece—such as a drum "riff" in a marching band or a jazz improvisation. In the case of a melody, the natural "pieces" are pitch/duration events (notes) and possible relations are the pitch/time intervals between notes. In the case of a drum piece, one might call the "pieces" "sound onsets" since that is actually the perceptually most relevant element. The relations, then, would be "durations," that is, the time between onsets. Some relations might be regarded as pieces in their own right (e.g., durations), and hence relations of relations (see below) might be considered.

In school and professional mathematics, a typical "thing" might be a geometric shape or construction, the pieces might be points or lines, and the relations among pieces could be distances between points or angles between line segments.

To make our mathematics, we need one more kind of thing: We need "transformations" that map one thing onto another. Thus, we might consider the transformation that maps one instance of a given melody onto another, one instance of a given drum piece onto another, or one instance of a given geometric figure onto another. We presume that the mapping "induces" submappings among the pieces, that is, we can identify the notes in the transformed melody that correspond to (map from) the notes in the original melody, or which points in the transformed shape correspond to which points in the original one.[7] If we do have the mapping between pieces, then we can ask whether corresponding relations are the same, before and after the transformation. When corresponding relations are the same, this is called *invariance*.

Mathematicians might be happier with some further specification and notation. But again we wish to emphasize that we are now moving into a notational realm that may refer to different kinds of entities and relations, and certainly operates at a different level of generalization. In particular, we will denote pieces in the original thing by x, y, and z, and corresponding pieces in the transformed thing by x′, y′, and z′. Obviously, we are assuming x "maps onto" x′, and so on. Relations may be represented in terms of units of measure, that is, relations map a pair of objects into numbers (or into similar "measure spaces," such as conventional ways of measuring intervals, which includes terms such as "major third," "diminished fifth," etc.).[8] If we denote a relation by R and a transformation by T, then the most important question we can ask is whether T preserves R, that is, whether R(x, y) = R(x′, y′). [The latter is by definition R(T(x), T(y)).] In alternative language R is an invariant of transformation T when, in general, R(x, y) = R(x′, y′).

Now, the set of all possible transformations is huge, and many of them will be functionally irrelevant. That is, we won't be able to see or hear the relationship between the original thing and its transformed version.[9] At the other extreme, a

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[7] In many cases, it might be more natural to think of the mapping as defined on the pieces, e.g., points get transformed into new points, notes get transformed to new notes, which induces a map from all aspects of the original melody to corresponding aspects of the transformed melody. Further, a transformation on a single dimension—for example, pitch—can induce a transformation on compound entities (a note includes both pitch and duration), and thus, on the whole "thing." Incidentally, such mappings would not be possible with the medieval neumes notation given the total lack of specificity with respect to properties—neither pitch nor duration. The fact that properties are not explicit prevents the mapping invariants across instances.

[8] Typical of mathematicians, "relations" might be more exotic things like "binary predicates," that map a pair of entities into "true" or "false." A familiar binary predicate would be "intersects." $Intersect(l_1, l_2) = true$ expresses the fact that the two lines, l_1 and l_2, intersect. A mathematician would not be uncomfortable with "relations of arity one," that map a single entity into a measure space, for example, we consider a "relation" to be merely the "size of a geometric element."

[9] This applies to some of the transformations that were particularly attractive to composers in the Renaissance period and currently composers with a more purely formal bent along with those doing

transformation that preserves everything can be utterly boring; it is a literal repetition. In between, we can classify transformations by what relations (or properties) they preserve and what relations (or properties) they don't preserve. That is, we can name invariants (and non-invariants) of the transformation. In music, we can further ask about the function of the non-invariants: what does the variation do?

Let us illustrate. In a short percussion solo, an element might be duration, the length of time between onsets, that is, between "hits" of the drum. This can be seen as an analog to the length of a segment or the distance between two points in geometry. An obvious relation is the ratio between durations. In simple cases, this is just the count of the number of shorter durations one can perceive in the longer one, which is precisely Leibnitz's implicit counting. If these relations, ratios of durations, are preserved, we perceive the rhythm as being "the same," only with a different tempo. We could say "the proportional structure of durations is invariant under the transformation of 'playing the same percussion piece' at a different tempo."

With respect to melodies, a transposition in the strictest sense preserves the relation, "pitch interval," between events (as well as retaining the relative durations), in which case we hear it as "the same tune." But music allows more subtle invariants that stretch our ability to hear "the same," while allowing variations that increase interest or serve more particular function for a composer. So, for example, a composer might chose to write "the same melodic pattern" (contour or shape), but shifted up or down along a given scale—as Vivaldi and others did and as in the two tuneblocks shown in Figure 21. Since the pitch intervals within a scale are not equal, in shifting a pattern up or down the scale, the general contour of the melodic segment remains invariant and the segment continues to be recognizable, even though the intervals between events are not exactly the same.

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algorithmic composition. These include, for instance, pitch transformations such as retrograde, where the succession of pitches is played backwards, and inversion where the succession of intervals in a melody are tipped upside down. It is usually quite difficult if not impossible actually to hear these transformations despite their attractiveness as apparent means of generating structural coherence. Haydn (1773), Bach (1781), and Schoenberg (1921) all used such transformations. We wish to distinguish these transformations from ones that are perceptually salient, such as those the children noticed.

More radically, a composer might write "the same melody" but using a pitch collection or scale that includes different internal intervals. For instance, she might shift from the seven pitches of a major scale collection, which underlies many popular and simple tunes in Western cultures, to the seven pitches of a minor scale collection, which have a slightly different set of internal interval relations. This gives the tune a different quality; a "happy" tune might become "sad." The invariant, here, is no longer pitch interval in a strict sense, since minor scales involve different basic intervals; instead, the invariant is a looser one, scale step or scale degree number in the ordered series of whatever scale is used. For instance, Figure 22 shows the tune, *Twinkle Twinkle Little Star* , first using the Major scale pitch collection and associated scale degrees, and then, using the minor scale pitch collection where the pitches are not exactly the same but the scale degrees remain invariant. Schubert often uses this ploy in songs to reflect a change of mood in a corresponding text.

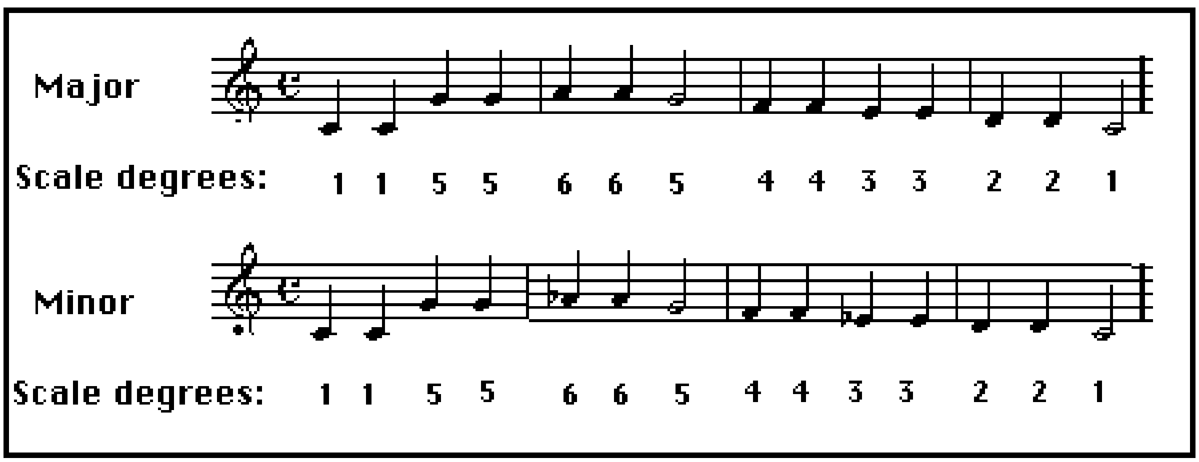

Figure 22. Star: Major and Minor.

Musically, we see one salient possible aesthetic game. How far can we press the transformation, and how little can remain invariant, before the relation is perceptually lost? Bach certainly played this game in the famously intractable "Goldberg Variations." Further, composers who invent new ways to change things that still preserve a sense of coherence (like Vivaldi's trademark sequences or John Coletrane's riffs), or who find new uses for the non-invariants, get credit for their invention.

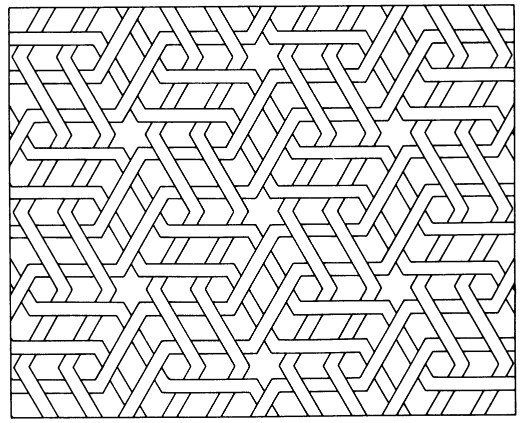

Figure 23. Left, an ornamental relief from the Alhambra palace involves multiple elements and variations. Right, a geometric pattern inspired by Islamic art involves a single element repeated in multiple orientations. The base element is difficult to find because it is, in fact, ambiguous, and because perceived continuations from one instance to the next suggest larger units. See Abelson & diSessa (1981), p. 103.

More of the aesthetic and mathematical games involved in transformations and invariants can be seen in Figure 23. Both images rely on transformations and invariants. Both produce global effects that transcend literal repetition, where elements are transformed in their effect by the local context. The design on the right is particularly clear in this regard because it is difficult even to see the repeated element; the eye combines it with its neighbors. The design on the left also uses literal repetition, or very simple transformations (mirror image), to good effect: There is a left-right symmetry in the picture. But there are more subtle transformations and invariants that are not easily captured in geometry. A trefoil of leaves appears here; a hextet of leaves (or is it a flower?) appears elsewhere. Is the image about organic forms; can we say "organic nature" is the invariant of many or most elements? Are the inscribed elements that background the trefoil "stylized leaves" or organic at all? Are they deliberately ambiguous?

The musical analog of many of these phenomena will play a role in the next section. For now, we position these observations with respect to the core issues of this paper. (1) Students manifestly hear certain kinds of invariance and can even appreciate the mathematical formulations of some of them (ratios of durations). How far can the *mathematics of invariance* be drawn out of musical experience? Does it take pre-instruction of the mathematics, or can it literally be drawn out of music (and/or visual art)? (2) Can that mathematics become a "language for design," that permits students to compose more effectively? In this regard, one would like to extend Impromptu with a language of motivic transformation and composition (as in, literally, putting together), so that students can explore transformation and invariance instrumentally, in creating music. What are the appropriate representational forms to make this possible, and to optimize mathematical relevance without usurping musical sense?[10]

In the final main section of this paper, we return to structure that we have evidence is inherent in students' perception of music, and to the strategy of telling the story from the musical (as opposed to mathematical) end. We now pick up chronologically from where we left off in recounting our experiences with a sixth grade group of children. We will continue to explore transformations and invariants, and their perceptual consequences, although we will not realize the thought experiment, above, to see whether the children can actually articulate the mathematics. In addition, we will look at the overall structure of a tune produced by transformations and variations, and how that structure is perceived.

THE STRUCTURE OF MELODIES

Searching for patterns had been very productive, but would the children use what they had heard and seen in composing their own tunes? To find out, the goal of the next project was: "Make a tune that makes sense and that you like using the ELI blocks."

To compose a tune, each of the children at his/her computer, listened to the ELI blocks, then dragging tuneblocks icons into the playroom area, they experimented with arranging and rearranging them as they listened for the results of their orderings.

After about 20 minutes of concentrated work, most of the children had completed a tune. Kathy's tune is shown as Figure 24. The blocked numbers are the numbers of the tuneblocks as shown in Table 1. The smaller numbers above the staff are measure numbers for reference in discussing the tune.

[10] The predecessor of Impromptu, MusicLogo, had many of these properties and facilitated some of the explorations suggested here.

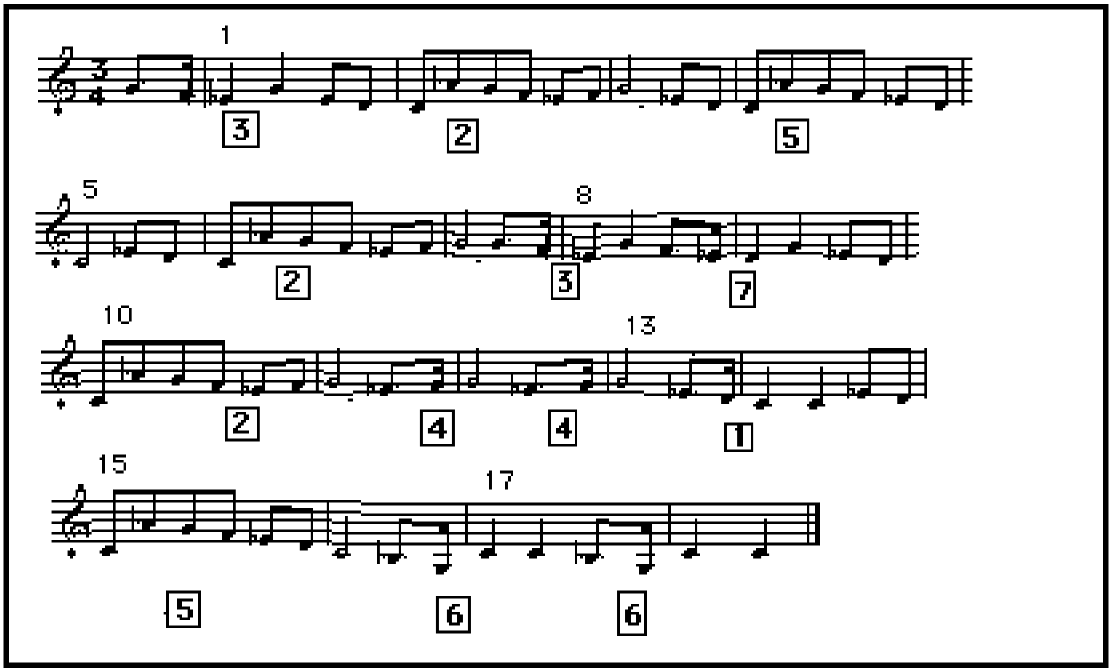

Figure 24. Kathy's tune.

Kathy used the patterns we had discussed in interesting ways. For instance, after a brief introduction using Block 3, in measures 2-5 she uses Tuneblocks **2** and **5** to form an antecedent/consequent pair—that is, two phrases that begin the same but end differently. Recall that Kathy heard blocks **2** and **5** as the same except for the last two notes. In her tune, Block **5** which sounded like an "ending," brings the previous Block **2** to rest.

Blocks **3** and **7** were described by the children as a "shoved down" version of one another—a sequential relationship (Figure 21).[11] Kathy uses the two blocks as a sequential pair in measures 8 and 9. The sequence is one of our examples of transformation and invariance—in this case a pattern of change analogous to the pattern of change, 1-3-5, that Kathy mentioned earlier.

Bars 10-11 includes Block **2**, which is followed in bars 12-13 by repetitions of Block **4**. Block **4** was described by the children as "a piece of Block **2**"—this is an example of fragmentation. The fragment, Block **4**, is repeated resulting in a kind of stretching of Block **2** while at the same time quickening the event-time because of

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[11] It is interesting that embedded in this context Block **3** is hardly recognizable as the same block with which her whole melody began. We hear it here, as a kind of continuation and variation of the preceding Block **2**, and, in retrospect, also the beginning of the upcoming sequence.

the shortening of the initial Block **4** (Figure 25). "Quickening" is often a useful function, to add tension, drama, contrast, etc.

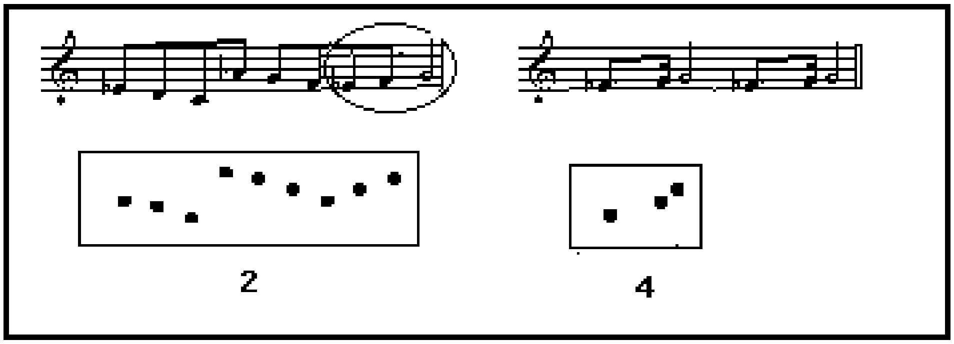

Figure 25. Fragmenting, stretching, quickening.

To finish off her melody, Kathy makes a "coda"—an extended "tail" (coda means "tail" in Italian). Juxtaposing Blocks **1** and **5,** in bars 14-16, the melody arrives three times at the most stable sounding pitch—the tonic (the tonal "home base," C). Then, prolonging that stability, in the final two measures Kathy repeats Block **6**, which keeps returning to this same tonic pitch (Figure 26). Following the two previous blocks, each of which ends solidly on the tonic in C minor, Block **6** brings the tune to a close with a kind of poignant sigh.[13]

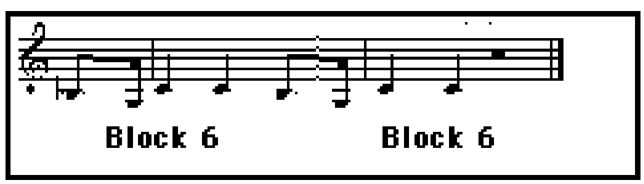

Figure 26. A Coda.

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[13] How do we abstract "a sigh" to see this last Block as one (playing a subtler version of the game we did earlier, with "faster")? A sigh might be described as a weakened after-comment. Notice that Block **6** ends on a weak beat after the strong beat ending of Block **5**. Block **6** is "after" in the sense that it is later, but also, strictly speaking, it is unnecessary. The piece has already been brought home to the tonic. The "weakened" part of this sketch might be emphasized in performance by reduced volume and/or slowed tempo, possibly deliberately separated somewhat rhythmically from the preceding segment.

Musical know-how

From the view of the children's intuitive musical know-how, there is no doubt that Kathy, as well as the other children, in composing their tunes are actually making use of structural relations that we have pointed to in our previous comments. These include, rule-driven transformations such as sequence, fragmentation, and extension by repetition. In addition there is no doubt that the children are able to hear the pitch that sounds most stable—that is, the tonic function. All of the children ended their tunes with either Block **1**, **5** or **6**, each of which ends on the tonic. Moreover, like Kathy, several other children ended their tunes with a tonic prolongation, most often the repeated Block **6**. Later on, three children from a new group, edited Block **6** by removing the last note (C), thus reinforcing the stability of the tonic by ending their tune on a more stable strong beat, as well. All of which seems evidence that the children have available *in-action* what Meyer calls musical "archetypes":

> [A]rchetypes may play a significant role in shaping aesthetic experience and fostering cultural continuity in the absence of any conscious conceptualization about their existence, nature, or kinds. Rather, they may be and usually are internalized as habits of perception and cognition operating within a set of cultural constraints. (Rosner & Meyer, 1982: 318)

CONCLUSION

Implications for Music Learning and Technology

As anticipated, the children's work provides provocative initial evidence for affinities between musical and mathematical structures. In addition, it provides an initial roadmap of particular important connections that might be made, and even fragments of interchange and inquiry where children seem to be building or at least capitalizing on the connections. The children seemed to gain insights and to move toward evocative generalizations through discoveries that rhythmic structures embody and also inform mathematical structures such as ratio, proportion, fractions, and common multiples. Similarly, the more general theme of transformations and invariants seems emergent and ready to be capitalized on. However, while we may see that affinity, it clearly remains as yet unrealized in the students' work.

The theoretical theme underlying this work is the complex set of relations among ontology, perceived experience, representations, dimensions, and formalized versions of structures "evident" in how children perceive and operate in a musical context. We emphasized the limited capability of "professional" representations to connect to experience, but also (in music) their limited capability to connect to generalization, beyond music. We've seen multiple representations and modalities exposing and helping to resolve paradoxes of perception and representation. Various representations and modalities arguably also help stabilize and make accessible dimensions for further consideration elements (e.g., pitch, duration; formal similarity like transposition and fragmentation) that are manifestly part of, but not the entirety of musical experience.

Perhaps the most general aspect of the affinity between mathematics and music might be *the perception and articulate study of patterns*. Pursuing this agenda within music might encourage children to become intrigued with looking for patterns in other domains as well. And it might lend a "sense" to mathematics as a tool for understanding more about what we intuitively have some grasp of and care about. Some of the simplest patterns become intriguing and paradoxical in a musical context. Consider repetition, which we unflatteringly characterized as "boring" above. Yet, even repetition is functional and can be an event full of subtlety. As one child said, on being asked to find repetition in a melody, "But it can never be the same because it's *later*." Indeed, a repeated melodic or even rhythm segment often sounds different and may function differently when embedded in a different context (as in Kathy's tune). With his focus specifically on rhythm, Christopher Hasty puts it this way:

> As something experienced, rhythm shares the irreducibility and the unrepeatability of experience... when it is past, the rhythmic event cannot be again made present... Rhythm is in this way evanescent: it can be "grasped" but not held fast. [Hasty, 1997: p. 12]

Perhaps in this sense, mathematics and music diverge—mathematics seeks to "hold fast" ideas that may be fleeting, while in actually perceiving music, we can say as Aristotle says of time:

> One part of it has been and is not, while the other is going to be and is not yet.... The "now" which seems to be bound to the past and the future—does it always remain one and the same or is it always other and other? It is hard to say. [Aristotle, *Physics,* p. 297-8]

It is worth underscoring what led to the productive emergence of affinities—and also to interest-spurring paradoxes and "contradictory" interpretations—in the experience of these students. Certainly it is rich intuitive knowledge, to begin. Unless students are sensitive to certain structures and patterns, there seems little basis on which to build. But students became *more sensitive*, and articulately so, to these patterns. So, they could, for example, make rhythmic accompaniment boring, or chaotic in a systematic way. Much was clearly gained by providing the possibility for children to move easily across media and sensory modalities, to have access to multiple kinds and levels of representations, and actually to make music building on their advancing ways of perceiving and conceiving it.

While the empirical work explored here involved only a small group of 6th grade children over a relatively brief period of time, the results suggest not only significant intersections between musical and mathematical conceptual structures, but also more general directions for the development of effective computer environments for learning.

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