The Gomory-Chvátal Closure: Polyhedrality, Complexity, and Extensions

by

Juliane Dunkel

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Abstract

In this thesis, we examine theoretical aspects of the Gomory-Chvátal closure of polyhedra. A Gomory-Chvátal cutting plane for a polyhedron P is derived from any rational inequality that is valid for P by shifting the boundary of the associated half-space towards the polyhedron until it intersects an integer point. The Gomory-Chvátal closure of P is the intersection of all half-spaces defined by its Gomory-Chvátal cuts.

While it is was known that the separation problem for the Gomory-Chvátal closure of a rational polyhedron is NP-hard, we show that this remains true for the family of Gomory-Chvátal cuts for which all coefficients are either 0 or 1. Several combinatorially derived cutting planes belong to this class. Furthermore, as the hyperplanes associated with these cuts have very dense and symmetric lattices of integer points, these cutting planes are in some sense the "simplest" cuts in the set of all Gomory-Chvátal cuts.

In the second part of this thesis, we answer a question raised by Schrijver (1980) and show that the Gomory-Chvátal closure of any non-rational polytope is a polytope. Schrijver (1980) had established the polyhedrality of the Gomory-Chvátal closure for rational polyhedra. In essence, his proof relies on the fact that the set of integer points in a rational polyhedral cone is generated by a finite subset of these points. This is not true for non-rational polyhedral cones. Hence, we develop a completely different proof technique to show that the Gomory-Chvátal closure of a non-rational polytope can be described by a finite set of Gomory-Chvátal cuts. Our proof is geometrically motivated and applies classic results from polyhedral theory and the geometry of numbers.

Last, we introduce a natural modification of Gomory-Chvátal cutting planes for the important class of 0/1 integer programming problems. If the hyperplane associated with a Gomory-Chvátal cut for a polytope $P \subseteq [0,1]^n$ does not contain any 0/1 point, shifting the hyperplane further towards P until it intersects a 0/1 point guarantees that the resulting half-space contains all feasible solutions. We formalize this observation and introduce the class of *M-cuts* that arises by strengthening the family of Gomory-Chvátal cuts in this way. We study the polyhedral properties of the resulting closure, its complexity, and the associated cutting plane procedure.

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Biographical Note

Juliane Dunkel received a Diploma in Civil Engineering in April 2004 and a Diploma in Mathematics in August 2005, both from the Berlin University of Technology in Germany. Starting in April 2004, she spent one year as a visiting student at the Operations Research Center at MIT to write her diploma thesis for the degree in Mathematics. In September 2005, she joined the Operations Research Center at MIT to pursue a Ph.D. During this time, she worked as a research intern at the IBM Research Center in Zurich from September 2007 until April 2008.

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Cambridge, March 2011 *Juliane Dunkel*

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Chapter 1

Introduction

The theory of integer linear programming provides some of the most important tools and approaches for solving combinatorial optimization problems. Cutting plane methods in combination with branch-and-cut algorithms are among the most successful techniques for solving integer linear programs. Hence, studying theoretical and computational aspects of cutting planes and understanding the structural properties of sets that naturally arise from them is fundamental in developing efficient algorithms for many important combinatorial optimization problems.

Integer linear programming is concerned with the optimization of a linear function over the integer points in a polyhedron. Formally, an integer linear program can be written as $\max\{cx \mid Ax \le b,\ x \in \mathbb{Z}^n\}$, where $c \in \mathbb{Z}^n$ is the objective function vector, $A \in \mathbb{Z}^{m\times n}$ the constraint matrix, and $b \in \mathbb{Z}^m$ the right-hand side vector. Integer programming is, in general, NP-hard and, hence, it is unlikely that there exists a polynomial-time algorithm that solves all instances. If $P = \{x \in \mathbb{R}^n \mid Ax \le b\}$ denotes the underlying polyhedron of the integer program and $P_I = \operatorname{conv}\{x \in \mathbb{Z}^n \mid x \in P\}$ the convex hull of all integer points in P, the integer program can be equivalently expressed as linear programming problem $\max\{cx \mid x \in P_I\}$. The difficulty in applying this equivalence is that representing P_I by linear inequalities is, generally, a difficult task. Given that a nice description of the integer hull P_I is known – in the sense that it can be decided in polynomial time whether a given point is contained in P_I or, if not, that a violated constraint can be exhibited in polynomial time – the problem can be solved efficiently via the ellipsoid method. However, for most combinatorial optimization problems a linear description of P_I with these properties is not known; and in the case that a problem is thought to be intractable, it is unlikely that such a description exists.

A general approach to solve these problems even so, is to approximate the convex hull of feasible integral solutions by some rational polyhedron P that has the same set of integer points and for which the *separation problem* can be decided efficiently. If the optimization over the relaxed polyhedron results in an integral optimal solution, the same point yields an optimal solution to the original problem. In general, however, one cannot assume this to be the case and cutting plane methods represent a powerful technique to iteratively arrive at an optimal integral solution. The basic idea behind these methods is to refine the relaxation P of the optimization problem by means of *cutting planes*. A cutting plane for a polyhedron P is an inequality that is valid for its integer hull P_I, but not necessarily for P itself. In other words, a cutting plane potentially strengthens the approximation P of P_I by cutting off fractional points in P. In a typical cutting plane algorithm, the optimization problem for the relaxation P is solved and, if the optimal solution x^* is fractional, a cutting plane that is violated by x^* is added to the system of inequalities defining P and, hence, a tighter approximation of P_I is obtained. In this manner valid inequalities are added to the current relaxation until the linear optimization results in an integral optimal solution.

Arguably, one of the most famous families of cutting planes was first introduced in Gomory's cutting plane method (Gomory 1958) and later, more formally studied, by Chvátal (1973). A *Gomory-Chvátal cutting plane* for a polyhedron P is an inequality of the form $ax \leq \lfloor a_0 \rfloor$, where a is an integral vector and $ax \leq a_0$ is valid for P. If the components of a are relatively prime, it is not difficult to see that $ax \leq \lfloor a_0 \rfloor$ describes the integer hull H_I of the half-space $H = \{x \in \mathbb{R}^n \mid ax \leq a_0\}$. Geometrically, H_I arises by shifting H towards the polyhedron until its bounding hyperplane contains an integer point. Clearly, for every half-space H with the property that $P \subseteq H$, it holds that $P_I \subseteq H_I$. Hence, the intersection of all half-spaces defined by the Gomory-Chvátal cuts of the polyhedron P defines an approximation of its integer hull P_I. The resulting set is called the *Gomory-Chvátal closure* of the polyhedron P or the *elementary closure*, and it is, typically, denoted by P'. Chvátal investigated the sequence of successively tighter approximations of the integer hull of a rational polyhedron that arises from a repeated application of all Gomory-Chvátal cuts to the polyhedron. This iterative process is also referred to as the *Gomory-Chvátal procedure*. The minimal number of rounds that are necessary to obtain P_I with this procedure is called the *Chvátal rank* of P. Chvátal showed that for every rational polytope this number is finite and Schrijver (1980) extended the result to rational polyhedra. Hence, the Gomory-Chvátal procedure defines a generic method to generate the integer hull of rational polyhedron, without requiring any knowledge of the structure of the underlying problem.

Given the formal definition of the elementary closure, it is not obvious whether the sets of successively tighter approximations obtained during the Gomory-Chvátal procedure are polyhedra themselves, that is, whether they can be described by a finite set of Gomory-Chvátal cuts. Yet, Schrijver (1980) showed that the Gomory-Chvátal closure of a rational polyhedron is itself a polyhedron: the Gomory-Chvátal cuts corresponding to a so-called *totally dual integral system* of linear inequalities determining P are sufficient to describe P'. In the same paper, Schrijver raised the question of whether the Gomory-Chvátal closure of a general polytope is also defined by a finite set of inequalities. As the notion of total dual integrality does not generalize towards non-rational systems of linear inequalities, the proof technique applied for the rational case does not extend to non-rational polyhedra. Moreover, it is well-known that the integer hull of a non-rational polyhedron may not be a polyhedron and the same is true for the Gomory-Chvátal closure of a non-rational polyhedron. However, for bounded polyhedra, that is, polytopes, the integer hull clearly is a polytope. The question as to whether the Gomory-Chvátal closure of a non-rational polytope is a polytope has remained open since Schrijver (1980) posted it. In this thesis, we answer this question in the affirmative and show that the Gomory-Chvátal closure of a non-rational polytope can be described by a finite set of Gomory-Chvátal cuts; that is, it is a polytope itself.

Most often integer programming formulations of combinatorial optimization problems have variables that can take the value 0 or 1, depending on the occurrence of a particular event. In other words, the feasible sets of the linear relaxations are polytopes in the unit cube $[0, 1]^n$. While one can already construct general polytopes of arbitrary large Chvátal ranks in two dimensions, the Chvátal rank of polytopes in the unit cube is polynomially bounded in the dimension (Bockmayr et al. 1999 and Eisenbrand and Schulz 2003). Hence, the special structure of these polytopes has a significant impact on the efficiency of the Gomory-Chvátal procedure. On the other hand, since Gomory-Chvátal cuts are defined for general integer programs, not all the structural properties that distinguish polytopes in the unit cube from the general case are exploited by the procedure. In this thesis, we introduce a natural strengthening of Gomory-Chvátal cutting planes for the important subclass of 0/1 integer programs. As described above, the classic Gomory-Chvátal cut associated with a valid inequality $ax \leq a_0$ is obtained by shifting the boundary of the half-space $H = \{x \in \mathbb{R}^n \mid ax \leq a_0\}$ towards the polytope until it intersects an integral point. The procedure guarantees that every integer point in $\mathbb{R}^n$ that satisfies the original inequality also satisfies the Gomory-Chvátal cut. However, to derive a cutting plane for a polytope P in the n-dimensional unit cube $[0,1]^n$, it suffices to require only that every point in $\{0,1\}^n$ that satisfies the original inequality satisfies the modified cut. This guarantees that no integer point in P violates the strengthened inequality. From a geometric point of view, we can push the boundary of H until it intersects a 0/1 point and, hence, we potentially obtain a valid inequality for the integer hull that strictly dominates the corresponding Gomory-Chvátal cut. In this thesis, we formalize this idea. We introduce a family of refined Gomory-Chvátal cuts for polytopes in the unit cube and study various aspects of the elementary closure arising from this class of cuts. For example with respect to structural properties, we show that for any rational polytope in the unit cube, the elementary closure associated with this family of refined Gomory-Chvátal cuts is again a polytope. Furthermore, we characterize its facet-defining inequalities and consider related complexity aspects. We also compare this new closure to elementary closures associated with important families of cutting planes for 0/1 integer programs discussed in the literature. Finally, we investigate the cutting plane procedure associated with these strengthened Gomory-Chvátal cuts. More precisely, we study how quickly an iterative application of the described family of cutting planes to a polytope generates its integer hull.

Given the significance of an efficient separation procedure when solving linear programming problems and the importance of cutting plane methods in integer programming, the meaning of efficient separation algorithms for families of cutting planes is evident. Eisenbrand (1999) proved that the problem of deciding whether a given point is contained in the Gomory-Chvátal closure of a polyhedron is NP-hard. His result is based on an observation of Caprara and Fischetti (1996a) who showed that the separation problem for the subclass of so-called $\{0, \frac{1}{2}\}$-Gomory-Chvátal cuts is NP-hard. In this thesis, we investigate the complexity of the separation problem for the subclass of

Gomory-Chvátal cuts $ax \leq \lfloor a_0 \rfloor$ for which the coefficients of the normal vector a are all either 0 or 1. On the one hand, several important combinatorially derived cutting planes belong to this family of cuts. On the other hand, the integral hyperplanes that are described by normal vectors in the set $\{0, 1\}^n$ are among the most symmetric hyperplanes spanned by integral points in $\mathbb{R}^n$. More precisely, the lattices of integer points in these hyperplanes are very dense. We show that for this subclass of Gomory-Chvátal cutting planes, the separation problem is already NP-hard.

1.1 Outline and Main Contributions

This thesis is divided into six chapters, including this chapter and the concluding Chapter 6. Each of the Chapters 3, 4, and 5 is largely coherent and builds on the results and concepts introduced in Chapter 2.

Chapter 2 While we assume that the reader is familiar with basic concepts of linear algebra, polyhedral theory, and complexity theory, we review in Chapter 2 the most important definitions and concepts that the subsequent chapters rely on. In particular, we introduce basic notations and provide an introduction into polyhedral theory, linear programming, integer programming, lattices, and cutting plane methods.

Chapter 3 In Chapter 3, we study the complexity of the membership and separation problem associated with the subclass of Gomory-Chvátal cuts $ax \leq \lfloor a_0 \rfloor$ for which the normal vector a has coefficients 0 or 1. Several combinatorially derived cutting planes belong to this class and the hyperplanes associated with these cuts are, in a sense, the most symmetric hyperplanes spanned by integer points. We show that the membership problem for the closure associated with this family of cutting planes is NP-hard and, hence, so is the corresponding separation problem.

Chapter 4 It has been well-known that, despite its definition as the intersection of an infinite number of integral half-spaces, the Gomory-Chvátal closure of any rational polyhedron is itself a polyhedron. A question that was raised by Schrijver (1980) and which has been open since then, is whether the same is true for the Gomory-Chvátal closure of a non-rational polytope. (For non-rational polyhedra, the Gomory-Chvátal closure does not need to be polyhedron.) In Chapter 4, we answer Schrijver's question and show that for every non-rational polytope a finite system of rational inequalities is sufficient to describe its Gomory-Chvátal closure. Hence, the Gomory-Chvátal closure of a non-rational polytope is, indeed, a polytope. Our proof is geometrically motivated and applies various classic results from polyhedral theory and the geometry of numbers.

Chapter 5 Gomory-Chvátal cutting planes can be applied to arbitrary integer programming problems, that is, arbitrary polyhedra in $\mathbb{R}^n$. As a result, their application to 0/1 integer programming problems cannot exploit all the structural properties that distinguish polytopes in the unit cube from general polyhedra. In Chapter 5, we therefore introduce a natural modification of Gomory-Chvátal cutting planes for the important subclass of polytopes in the unit cube $[0,1]^n$. This new family of cutting planes arises from the set of Gomory-Chvátal cuts by strengthening the right-hand sides of cuts $ax \leq \lfloor a_0 \rfloor$ for which the hyperplane $\{x \in \mathbb{R}^n \mid ax = \lfloor a_0 \rfloor\}$ does not contain any 0/1 point. Hence, in general, the implied cutting plane dominates the corresponding Gomory-Chvátal cut. As a result, the elementary closure of a polytope $P \subseteq [0,1]^n$ associated with this new family of cutting planes defines an approximation of the integer hull P_I that is potentially better than the Gomory-Chvátal closure P'. We study various aspects of this new closure and show that several properties of the Gomory-Chvátal closure have a natural analog with respect to the strengthened closure. Moreover, we compare this closure with other elementary closures from the literature, investigate the cutting plane procedure associated with it, and examine related complexity questions.

Chapter 2

Preliminaries

We assume that the reader is familiar with basic aspects of linear algebra, graph theory, convex and polyhedral theory as well as complexity theory. A nice introduction to polyhedral theory and proofs of many of the results presented in Sections 2.3 to 2.6 can be found in textbooks such as Schrijver (1986) or Ziegler (1995). For an introduction to the theory of computational complexity, we refer to Papadimitriou (1994) and Garey and Johnson (1979).

2.1 Basics and Notations

The symbols $\mathbb{R}, \mathbb{Q}, \mathbb{Z}$, and $\mathbb{N}$ denote the set of real, rational, integer, and natural numbers, respectively. $\mathbb{R}_+, \mathbb{Q}_+, \mathbb{Z}_+$, and $\mathbb{N}_+$ are the restrictions of these sets to the nonnegatives. If α is a real number, then $\lfloor \alpha \rfloor$ denotes the largest integer less than or equal to α and $\lceil \alpha \rceil$ denotes the smallest integer larger than or equal to α. For integer numbers a and b, we say that b divides a if $a = \lambda b$ for some integer λ. If $a_1, \ldots, a_n$ are integer numbers, not all equal to 0, then the largest integer dividing each of $a_1, \ldots, a_n$ is called the *greatest common divisor* of $a_1, \ldots, a_n$; it is denoted by $\gcd(a_1, \ldots, a_n)$. The numbers $a_1, \ldots, a_n$ are called *relatively prime* if $\gcd(a_1, \ldots, a_n) = 1$. For $1 \leq i \leq n$, we denote by e_i the unit vector in $\mathbb{R}^n$ that has all components 0, except for the i-th component, which is 1.

2.2 Linear Algebra

For sets U and V, we write $U \subseteq V$, if U is contained in V. If the containment is strict, we write $U \subset V$. If U and V are subsets of $\mathbb{R}^n$, then $U + V := \{u + v \mid u \in U, v \in V\}$. If $V = \{v\}$ is a singleton, then we write $U + v$ instead of $U + \{v\}$.

For a row vector $v = (v_1, \ldots, v_n)$, we use the same notation v to address its column vector v^T. We implicitly assume compatibility of sizes when we write products between vectors or products between vectors and matrices. Given two vectors $u, v \in \mathbb{R}^n$, we denote their scalar product by uv, that is $uv = \langle u, v\rangle = \sum_{i=1}^n u_i v_i$. If $uv = 0$, that is, the vectors u and v are perpendicular, then we also write $u \perp v$. We use the notations $\|v\|$ and $\|v\|_\infty$ for the *Euclidean norm* and the *maximum norm* of the vector v, respectively. That is, $\|v\| := \|v\|_2 := \sqrt{vv}$ and $\|v\|_\infty := \max\{|v_1|, \ldots, |v_n|\}$.

The full-dimensional *ball* centered at a point $c \in \mathbb{R}^n$ with radius ρ is denoted by $B(c, \rho)$, that is, $B(c, \rho) = \{x \in \mathbb{R}^n \mid \|x - c\| \leq \rho\}$.

Let S be a subset of the Euclidean space $\mathbb{R}^n$. A point $x \in \mathbb{R}^n$ is an *interior point* of S if there exists an $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq S$. The set of all interior points of S is called the *interior* of S and denoted by $\operatorname{int}(S)$. The (topological) *closure* of S, denoted by $\bar{S}$, is the set of points $x \in \mathbb{R}^n$ such that every open ball centered at x contains a point of S. The boundary $\operatorname{bd}(S)$ of S is the set of all points in the closure of S that are not interior points, that is, $\operatorname{bd}(S) = \bar{S} \setminus \operatorname{int}(S)$. There are natural refinements of the concepts interior and boundary for a lower-dimensional set S, when S is considered as a subset of its affine hull. The *affine hull* of S, denoted by $\operatorname{aff}(S)$, is the smallest affine set containing S. That is,

$$\operatorname{aff}(S) := \left\{ \sum_{i=1}^{k} \alpha_i x_i \;\middle|\; x_i \in S, \alpha_i \in \mathbb{R}, i = 1, \ldots, k;\ \sum_{i=1}^{k} \alpha_i = 1;\ k \in \mathbb{N},\ k > 0 \right\} .$$

The *relative interior* $\operatorname{ri}(S)$ is defined as

$$\operatorname{ri}(S) := \{x \in S \mid \exists \varepsilon > 0, B(x, \varepsilon) \cap \operatorname{aff}(S) \subseteq S\} ,$$

and the *relative boundary* is the set $\operatorname{rbd}(S) := \bar{S} \setminus \operatorname{ri}(S)$. The set S is *convex*, if for any two points $x, y \in S$ and any $\lambda \in [0, 1]$, $\lambda x + (1 - \lambda) y \in S$. The smallest convex set containing S is called the *convex hull* of S and denoted by $\mathrm{conv}(S)$. We have

$$\mathrm{conv}(S) := \left\{ \sum_{i=1}^{k} \alpha_i x_i \,\middle|\, x_i \in S, \alpha_i \in \mathbb{R}_+, i = 1, \ldots, k;\ \sum_{i=1}^{k} \alpha_i = 1;\ k \in \mathbb{N},\ k > 0 \right\} .$$

The *conical hull* of S is the set of all *conical combinations* of vectors in S, that is

$$cone(S) := \left\{ \sum_{i=1}^{k} \alpha_i x_i \,\middle|\, x_i \in S, \alpha_i \in \mathbb{R}_+, i = 1, \ldots, k;\ k = 1, 2, \ldots \right\} .$$

For an affine space $H \subseteq \mathbb{R}^n$, the *maximal rational affine subspace* of H is the affine hull of the set of rational points in H.

2.2.1 Unimodular Transformations

A matrix $U \in \mathbb{Z}^{n \times n}$ is called *unimodular* if it is invertible and $U^{-1} \in \mathbb{Z}^{n \times n}$. In particular, $U \in \mathbb{Z}^{n \times n}$ is unimodular if and only if $\det(U) = \pm 1$. Every unimodular matrix can be obtained from the identity matrix by elementary column operations. Hence, unimodular matrices describe series of elementary column operations.

A map $f : \mathbb{R}^n \mapsto \mathbb{R}^k$ is an *affine map* if $f(x) = Lx + t$ for some matrix $L \in \mathbb{R}^{k \times n}$ and vector $t \in \mathbb{R}^k$. A *unimodular transformation* is an affine map $u : \mathbb{R}^n \mapsto \mathbb{R}^n$ with $u(x) = Ux + v$ such that $U \in \mathbb{Z}^{n \times n}$ is a unimodular matrix and $v \in \mathbb{Z}^n$. Note that a unimodular transformation is a bijection of $\mathbb{Z}^n$.

2.2.2 Orthogonal Projections

Let $U \subseteq \mathbb{R}^n$ be a linear vector space. The *orthogonal complement* $U^\perp$ of U is the set of all vectors that are orthogonal to every vector in U, that is,

$$U^\perp := \{ v \in \mathbb{R}^n \mid v \perp u \text{ for all } u \in U \} .$$

Given two vectors $u, v \in \mathbb{R}^n$, the vector v can be uniquely decomposed as $v = v_u + \tilde{v}$, where $v_u = \alpha u$ for some $\alpha \in \mathbb{R}$ and where $\tilde{v}$ is a vector orthogonal to u. The vector v_u

is called the *orthogonal projection of v onto u* and given by

$$v_u = \mathrm{proj}_u(v) := \frac{uv}{uu}\, u \ .$$

Similarly, if U is a linear vector space and $v \in \mathbb{R}^n$, then v can be written as $v = u + \tilde{v}$, where $u \in U$ and $\tilde{v} \in U^\perp$. Then u is the *orthogonal projection of v onto U* and $\tilde{v}$ is the orthogonal projection of v onto $U^\perp$. In other words, $\tilde{v}$ is the unique vector satisfying

$$\tilde{v} \in U^\perp \qquad \text{and} \qquad (v - \tilde{v}) \in U \ .$$

The length of the vector $\tilde{v}$ is called the *distance* between the v and the vector space U, it is denoted by $\mathrm{dist}(v, U) := \|\tilde{v}\|$. If U is a k-dimensional linear vector space with orthogonal basis $u_1, \ldots, u_k$, then $v = u + \tilde{v}$ with

$$\tilde{v} = v - \sum_{j=1}^{k} \mathrm{proj}_{u_j}(v) = v - \sum_{j=1}^{k} \frac{vu_j}{u_j u_j}\, u_j \ .$$

2.2.3 Parallelepipeds and Volume

Let $u_1, \ldots, u_k$ be linearly independent vectors in $\mathbb{R}^n$. The *parallelepiped* spanned by these vectors is the set

$$\overline{\Pi}(u_1, \ldots, u_k) := \left\{ \sum_{i=1}^{k} \lambda_i u_i \,\middle|\, 0 \le \lambda_i \le 1,\ i = 1, \ldots, k \right\} \ .$$

The corresponding *semi-open parallelepiped* is the set

$$\Pi(u_1, \ldots, u_k) := \left\{ \sum_{i=1}^{k} \lambda_i u_i \,\middle|\, 0 \le \lambda_i < 1,\ i = 1, \ldots, k \right\} \ .$$

Let A denote the matrix with columns $u_1, \ldots, u_k$. If $k = n$, then A is a non-singular square matrix and the absolute value of the determinant of A equals the volume of the parallelepiped spanned by $u_1, \ldots, u_n$. More precisely, let $\tilde{u}_1 := u_1$ and let $\tilde{u}_i$ denote the orthogonal projection of u_i onto $(\text{span}(u_1, \ldots, u_{i-1}))^\perp$, for $i = 2, \ldots, n$. Then

$$|\det(A)| = \text{vol}\Big(\Pi(u_1, \ldots, u_n)\Big) = \prod_{i=1}^{n} \|\tilde{u}_i\| \quad . \tag{2.1}$$

In general, if $k < n$, then the k-dimensional volume of the parallelepiped spanned by $u_1, \ldots, u_k$ is given by

$$\sqrt{|\det(A^T A)|} = \text{vol}\Big(\Pi(u_1, \ldots, u_k)\Big) = \prod_{i=1}^{k} \|\tilde{u}_i\| \quad . \tag{2.2}$$

2.3 Polyhedra and Linear Inequalities

A *polyhedron* in $\mathbb{R}^n$ is a set of the form $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, where A is a real matrix and b a real vector. The matrix A is called the *constraint matrix* of P. If A and b can be chosen such that all their entries are rational, then P is a *rational polyhedron*. A bounded polyhedron is called a *polytope*. The polyhedron $C = \{x \in \mathbb{R}^n \mid Ax \leq 0\}$ is called a *polyhedral cone*; if $0 \in C$, then the cone is *pointed*. The *dimension* of a polyhedron P equals $(|S| - 1)$, where S is a maximal affinely independent subset of P; that is, $\dim(P) = \dim(\text{aff}(P))$. A polyhedron $P \subseteq \mathbb{R}^n$ is said to be *full-dimensional* if it has dimension n.

From a geometrical point of view, a polyhedron is the intersection of a finite number of *affine half-spaces*, where an affine half-space is a set of the form $H = \{x \in \mathbb{R}^n \mid ax \leq a_0\}$ for some non-zero vector a and some number a_0. We write the half-space H also as $(ax \leq a_0)$. The hyperplane $\{x \in \mathbb{R}^n \mid ax = a_0\}$ is the boundary of the half-space H and denoted by $(ax = a_0)$. If H is a *rational half-space*, that is, $H = (ax \leq a_0)$ for some $(a, a_0) \in \mathbb{Q}^{n+1}$, then it always has a representation in which a is an integral vector with relatively prime components. If H cannot be described by rational data, it is a *non-rational* half-space. Similarly, the hyperplane $(ax = a_0)$ is a *non-rational hyperplane* if $(ax \leq a_0)$ is a non-rational half-space. We say that an inequality $ax \leq a_0$ is a *non-rational inequality* if the associated half-space $(ax \leq a_0)$ is non-rational.

An equivalent definition of polyhedra says that every polyhedron is the sum of a convex hull of a finite set of points plus a conical combination of a finite set of vectors. This fact is summarized in the following theorem (see also Corollary 7.1b in Schrijver 1986).

Theorem 2.1 ***(Decomposition theorem for polyhedra)*** *A subset $P \subseteq \mathbb{R}^n$ is a polyhedron, that is, an intersection of a finite set of closed half-spaces $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, if and only if there exist vectors $v^1, \dots, v^s$ and $r^1, \dots, r^t$ in $\mathbb{R}^n$ such that*

$$P = \operatorname{conv}(v^1, \dots, v^s) + \operatorname{cone}(r^1, \dots, r^t) \ .$$

Note that the Decomposition theorem holds both in real and rational spaces. In the latter case, all vectors are restricted to the rationals. Hence, every polyhedron is finitely generated and every finitely generated set is a polyhedron. In particular, a polytope is the convex hull of a finite number of points.

The *integer hull* P_I of a polyhedron P is the convex hull of the integer points in P, that is, $P_I = \operatorname{conv}\bigl(\{x \mid x \in P \cap \mathbb{Z}^n\}\bigr)$. The following important observation is due to Meyer (1974):

Theorem 2.2 *If P is a rational polyhedron, then P_I is a rational polyhedron.*

If $P = P_I$, then P is called an *integral polyhedron*. In particular, an integral polyhedron is the convex hull of its integral points. A *0/1 polytope* is the convex hull of a subset of $\{0, 1\}^n$.

We say that a linear inequality $ax \leq a_0$ is *valid* for a polyhedron P if it is satisfied by all points in $x \in P$. A valid inequality $ax \leq a_0$ is called a *supporting hyperplane* of P if $a\bar{x} = a_0$ for some $\bar{x} \in P$. A *face* of a polyhedron P is a non-empty set of the form

$$F = P \cap (ax = a_0) \ ,$$

such that $ax \leq a_0$ is a valid inequality for P. In particular, P itself is a face of P. The inequality $ax \leq a_0$ is also called a *face-defining* inequality. We say that F is a *non-rational face*, if it cannot be written as $F = P \cap (ax = a_0)$ such that $(a, a_0) \in \mathbb{Q}^{n+1}$. The following equivalent characterization can be found in Schrijver (1980).

Proposition 2.3 *A set F is a face of a polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ if and only if F is non-empty and of the form $F = \{x \in P \mid A'x = b'\}$ for some subsystem $A'x \leq b'$ of $Ax \leq b$.*

Similar to the dimension of the polyhedron P, the dimension of a face F of P is defined as the dimension of its affine hull $\text{aff}(F)$. A face F is called *proper* if $F \subset P$. The faces of dimensions $0, 1$, and $\dim(P)-1$ are called *vertices, edges*, and *facets*, respectively. In particular the facets are the maximal proper faces. A face-defining inequality $ax \leq a_0$ such that $F = P \cap (ax = a_0)$ is a facet, is called a *facet-defining* inequality. A *minimal face* of P is a non-empty face that does not contain any other face. For a polytope, the vertices are the minimal faces. The following result is due to Hoffman and Kruskal (1956):

Theorem 2.4 *A set F is a minimal face of a polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ if and only if F is non-empty and*

$$F = \{x \in \mathbb{R}^n \mid A'x = b'\}$$

for some subsystem $A'x \leq b'$ of $Ax \leq b$.

A very useful and famous result regarding linear systems of inequalities is due to Farkas and appears in the literature in many equivalent forms. Here, we state an 'affine' version due to Haar (1918) and Weyl (1935).

Lemma 2.5 ***('affine' Farkas' lemma)*** *Let the system $Ax \leq b$ of linear inequalities have at least one solution and suppose that the linear inequality $ax \leq a_0$ holds for each x satisfying $Ax \leq b$. Then for some $a_0' \leq a_0$ the linear inequality $ax \leq a_0'$ is a nonnegative linear combination of the inequalities in the system $Ax \leq b$.*

2.4 Linear Programming

Linear programming is concerned with the maximization of a linear function over a polyhedron. The *linear programming problem* (LP) can be stated in many equivalent forms, one of them being

$$\begin{aligned} \max \quad & cx \\ \text{s.t.} \quad & Ax \leq b, \end{aligned} \tag{2.3}$$

where A is a rational matrix and b and c rational vectors of suitable dimensions. Equivalently, the problem can be written as $\max\{cx \mid x \in P\}$, where $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$. A point $x \in P$ is called a *feasible solution* of the LP and the set of all feasible solutions is the *feasible region*. A linear programming problem is *feasible*, if the feasible region is nonempty, and otherwise it is *infeasible*. An important result of linear programming is the following duality theorem by von Neumann (1974) and Gale, Kuhn, and Tucker (1951).

Theorem 2.6 *Let A be a matrix and let b and c be vectors. Then*

$$\max\{cx \mid Ax \leq b\} = \min\{by \mid y \geq 0,\ yA = c\} \ ,$$

provided that both sets are nonempty.

Arguably, the best-known algorithm for solving linear programming problems is the *simplex method* designed by Dantzig (1951). The basic idea of this combinatorial method is to "walk along" the edges of the polyhedron underlying the linear program from vertex to vertex, until an optimal vertex is reached. Although some artificial instances show exponential running time, in practice and on the average the method is very efficient. With Khachiyan's method (Khachiyan 1979), an extension of the ellipsoid method to linear programming, one can solve explicitly given linear programs in polynomial time, at least theoretically. Even if no explicit representation of a polyhedron P is known in terms of linear inequalities, there is an extension of the method to problems for which one can efficiently solve the following problem:

Definition 2.7 *The separation problem is:*

> *Given a polyhedron $P \subseteq \mathbb{R}^n$ and a vector $\bar{x} \in \mathbb{R}^n$, decide whether $\bar{x} \in P$ and, if not, find an inequality which is valid for P but violated by $\bar{x}$.*

The polynomial-time equivalence of separation and optimization was observed by Grötschel, Lovász, and Schrijver (1981) and provides a powerful tool for solving certain combinatorial optimization problems in polynomial time.

Theorem 2.8 *For any polyhedron $P \subseteq \mathbb{R}^n$, the separation problem is solvable in polynomial time if and only if the optimization problem is solvable in polynomial time.*

2.5 Integer Programming

If the set of feasible solutions of a linear program (2.3) is further constrained to integer points, the resulting problem is an *integer (linear) programming problem*:

$$\begin{aligned} \max \quad & cx \\ \text{s.t.} \quad & Ax \leq b \\ & x \text{ integral,} \end{aligned} \tag{2.4}$$

where A is a rational matrix and b is a rational vector. With $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ being the underlying polyhedron, we can write the problem as

$$\max \{cx \mid x \in P \cap \mathbb{Z}^n\} \quad . \tag{2.5}$$

The corresponding linear program $\max \{cx \mid x \in P\}$ is called the *linear programming relaxation*.

Unlike (2.3), the integer programming problem (2.4) has no known polynomial-time algorithm. The problem is NP-complete, which means it is unlikely that such an algorithm exists. As the integer hull P_I of any rational polyhedron P is also a rational polyhedron (see Theorem 2.2), the integer programming problem (2.4) can be reduced to the linear programming problem

$$\max \{cx \mid x \in P_I\} \quad .$$

However, there are difficulties related to this equivalence. First, there are many classes of polytopes P for which the number of inequalities that are necessary to describe P_I

is exponentially large in terms of the size of the inequalities describing P. In addition, describing the inequalities appears to be a difficult problem. Unless NP = co-NP, one cannot find for any NP-complete problem a so-called *NP-description* of a class of inequalities that describes all facets of the convex hull of integral solutions. Such a description essentially allows us to decide in polynomial time, whether an inequality belongs to the class or not (Karp and Papadimitriou 1980).

2.5.1 Hilbert Bases and Total Dual Integrality

A finite set of vectors $a_1, \dots, a_k \in \mathbb{R}^n$ is called a *Hilbert basis* of the polyhedral cone $C = \operatorname{cone}(a_1, \dots, a_k)$ if each integral vector in C can be written as a nonnegative integral linear combination of $a_1, \dots, a_k$. An *integral Hilbert basis* is a Hilbert basis that consists of integral vectors only. Every pointed rational polyhedral cone has a unique minimal integral Hilbert basis (minimal with respect to taking subsets). The following definition of *total dual integrality* is closely related to Hilbert bases and integral polyhedra.

Definition 2.9 *A system of rational linear inequalities $Ax \leq b$ is called totally dual integral if the minimum in*

$$\min\{yb \mid yA = w,\ y \geq 0\}$$

is attained by an integral vector y, for each integral vector w for which the minimum exists.

Linear duality (Theorem 2.6) implies that if, $Ax \leq b$ is a totally dual integral system and b integral, the polyhedron $\{x \in \mathbb{R}^n \mid Ax \leq b\}$ is integral. The following result is due to Giles and Pulleyblank (1979):

Theorem 2.10 *For each rational polyhedron $P \subseteq \mathbb{R}^n$ there exists a totally dual integral system $Ax \leq b$ such that A is integral and $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$.*

2.6 Integer Linear Algebra

We say that a $k \times n$ matrix of full row rank is in *Hermite normal form* if it has the form $[B\ 0]$, where $B \in \mathbb{R}^{k\times k}$ is a nonsingular, lower triangular, nonnegative matrix. Every matrix of full row rank can be brought into this form by a series of elementary column operations.

Theorem 2.11 ***(Hermite normal form)*** *If B is a $k \times n$ matrix of full row rank, then there exists a unimodular matrix $U \in \mathbb{Z}^{n\times n}$ such that BU is a matrix in Hermite normal form.*

The following corollary gives necessary and sufficient conditions for the feasibility of a single linear diophantine equation (see also Corollary 4.1a in Schrijver 1986).

Corollary 2.12 *Let $a_1, \ldots, a_n$ be integer numbers. Then the linear equation $a_1x_1 + \ldots + a_nx_n = \beta$ has an integral solution x if and only if β is an integral multiple of* $\gcd(a_1, \ldots, a_n)$.

In particular, if $a \in \mathbb{Z}^n$ is a vector with relatively prime components, then there exists an integral vector $h \in \mathbb{Z}^n$ such that $ah = 1$.

Given linearly independent vectors $b_1, \ldots, b_k$ in $\mathbb{R}^n$, the *lattice* $\Lambda(b_1, \ldots, b_k)$ generated by these vectors is the set of all integral linear combinations of $b_1, \ldots, b_k$, that is,

$$\Lambda(b_1, \ldots, b_k) := \left\{ \sum_{i=1}^{k} \lambda_i b_i \;\middle|\; \lambda_i \in \mathbb{Z},\ i = 1, \ldots, k \right\} .$$

The set of vectors $b_1, \ldots, b_k$ is called a *basis* and k is called the rank of the lattice. A lattice can have several bases:

Theorem 2.13 *Let Λ be a lattice in $\mathbb{R}^n$ with basis $b_1, \ldots, b_k$ and let $b'_1, \ldots, b'_k$ be points of Λ. Furthermore, let B and B' denote the matrices with columns $b_1, \ldots, b_k$ and $b'_1, \ldots, b'_k$, respectively. Then $b'_1, \ldots, b'_k$ is a basis of Λ if and only if there exists a unimodular matrix $U \in \mathbb{Z}^{k\times k}$ with $B' = BU$.*

It follows from the last theorem and (2.2) that the volume of the *fundamental parallelepiped* of a basis of the lattice Λ, that is, the volume of the parallelepiped spanned by the basis vectors, does not depend on the basis itself:

$$\sqrt{|\det((B')^T(B')|} = \sqrt{|\det((BU)^T(BU))|} = \sqrt{|\det(U^T(B^TB)U)|} = \sqrt{|\det(B^TB)|} \ .$$

The number $\sqrt{|\det(B^TB|}$ is called the *determinant* of the lattice and denoted by $\det(\Lambda)$. In particular, if $B = (b_1, \ldots, b_n)$ is a basis of Λ, then $\det(\Lambda) = |\det(B)|$. From a geometric point of view, it is clear that

$$\det(\Lambda) \leq \prod_{i=1}^{n} \|b_i\| \ .$$

The last inequality is also known as *Hadamard's inequality.* Hermite showed that every lattice Λ has a basis $b_1, \ldots, b_n$ such that

$$\prod_{i=1}^{n} \|b_i\| \leq c_n \det(\Lambda) \ , \tag{2.6}$$

where c_n is a constant only depending on n. Note that if $b_1, \ldots, b_n$ is an orthogonal basis, then $\|b_1\| \ldots \|b_n\| = \det(\Lambda)$. The ratio $(\|b_1\| \ldots \|b_n\|)/\det(\Lambda)$ is also called the *orthogonality defect* of the basis.

A famous algorithm due to Lenstra, Lenstra, and Lovász (1982) – the *lattice basis reduction algorithm* – constructs a basis with small orthogonality defect:

Theorem 2.14 *Let $\Lambda \subseteq \mathbb{R}^n$ be a lattice. Then there exists a basis $b_1, \ldots, b_n$ such that*

$$\prod_{i=1}^{n} \|b_i\| \leq 2^{n(n-1)/4} \det(\Lambda) \ .$$

Note that the constant in Theorem 2.14 is not the same as the one in (2.6), as the result by Lenstra, Lenstra, and Lovász is constructive. A basis that satisfies the property of Theorem 2.14 is called a *reduced basis.* The result naturally generalizes to lower-dimensional lattices. There is another connection between the determinant of a matrix and the parallelepiped spanned by its columns: the absolute value of the determinant of an integral non-singular square matrix equals the number of integer points that are contained in the parallelepiped spanned by the columns of the matrix (see Chapter 7, Corollary 2.6 in Barvinok 2002).

Lemma 2.15 *Let $b_1, \ldots, b_n \in \mathbb{Z}^n$ be linearly independent vectors. Then the number of integer points in the semi-open parallelepiped*

$$\left\{ \sum_{i=1}^{n} \lambda_i b_i \,\middle|\, 0 \leq \lambda_i < 1,\ i = 1, \ldots, n \right\}$$

is equal to the absolute value of the determinant of the matrix with columns $b_1, \ldots, b_n$.

2.7 The Cutting Plane Method

The *cutting plane method* was developed by Gomory (1958) for solving integer linear programs with the simplex method. A *cutting plane* for a polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ is an inequality that is valid for the integer hull P_I, but not necessarily for P. The general idea of the cutting plane method is to refine the feasible set of an optimization problem by means of cutting planes, thereby obtaining a better approximation of P_I. In a typical cutting plane algorithm, one starts with the linear relaxation P and solves the corresponding linear program. If the optimal solution x^* is fractional, a cutting plane that is violated by x^* is added to the system of inequalities defining P and, hence, a better approximation of P_I is obtained. In this manner, valid inequalities are added to the current relaxation until the linear optimization results in an integral optimal solution.

Over the years, numerous types and families of cutting planes have been introduced and studied in the literature. Some of these cutting planes were derived for particular combinatorial optimization problems and, hence, rely on a deeper knowledge of the specific underlying structure of a problem. On the other hand, there are *general* families of cutting planes that can be derived based solely on the polyhedral representation of the problem. For an overview on the latter, we refer the reader to Cornuéjols (2008) and Cornuéjols and Li (2000). In this thesis, we merely want to review the cutting planes that were generated by Gomory's cutting plane method (Gomory 1958) and more formally introduced by Chvátal (1973).

2.7.1 Gomory-Chvátal Cutting Planes and the Elementary Closure

The simplest type of a rational polyhedron is a rational half-space, that is, a polyhedron defined by a single rational inequality. Every rational half-space H has a representation $H = (ax \leq a_0)$, where a is a nonzero vector with relatively prime integer components. The integer hull of H is given by

$$H_I = (ax \leq \lfloor a_0 \rfloor) \ .$$

To see this, observe the following: The subspace of $\mathbb{R}^n$ defined by the system $(ax = 0)$ is integral because a is an integral vector. Furthermore, $\gcd(a_1, \ldots, a_n) = 1$ implies that there exists some integral vector y with $ay = 1$ (see Theorem 2.12). Then every hyperplane $(ax = a_0)$ can be seen as the translation of $(ax = 0)$ by the vector $a_0 y$. If $a_0 \in \mathbb{Z}$, then every integer point in $(ax = 0)$ is mapped to an integer point and, hence, $(ax = a_0)$ is integral. Geometrically, H_I arises from H by shifting the boundary of H inwards until it contains an integral vector. The inequality $ax \leq \lfloor a_0 \rfloor$ is a cutting plane for H, since it is satisfied by all integer points in the half-space.

If P is a polyhedron and H a rational half-space such that $P \subseteq H$, then clearly $P_I \subseteq H_I$. Consequently, the inequality $ax \leq \lfloor a_0 \rfloor$ is also a cutting plane of P. By considering all rational half-spaces H with $P \subseteq H$, we obtain the *elementary closure* of the polyhedron P:

$$P' := \bigcap_{(ax \leq a_0) \supseteq P,\ a \in \mathbb{Z}^n} (ax \leq \lfloor a_0 \rfloor) \tag{2.7}$$

Clearly, one can restrict the intersection in (2.7) to integral normal vectors with relatively prime components and half-spaces such that the bounding hyperplane $(ax = a_0)$ is a supporting hyperplane of P. While the set of inequalities in (2.7) is infinite, (Schrijver 1980) showed that a finite number of them are sufficient to describe P'.

Theorem 2.16 *Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ be a polyhedron such that $Ax \leq b$ is a totally dual integral system and A an integral matrix. Then $P' = \{x \in \mathbb{R}^n \mid Ax \leq \lfloor b \rfloor\}$. In particular, for any rational polyhedron P, the set P' is a polyhedron.*

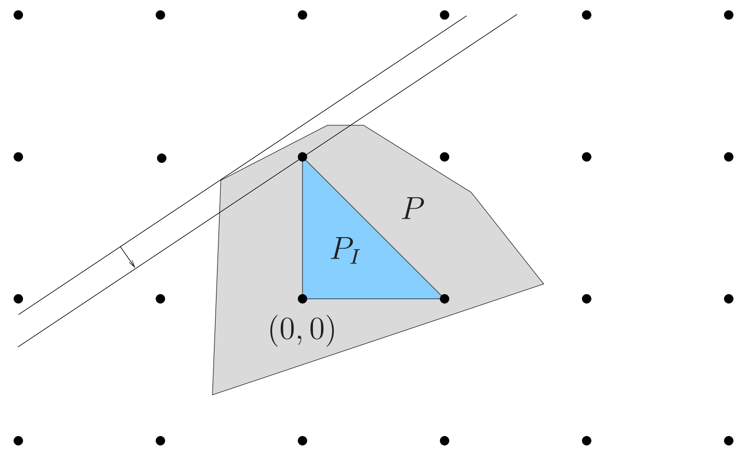

Figure 2-1: The inequality $-2x_1 + 3x_2 \leq 4 - \varepsilon$ is valid for P for some $\varepsilon > 0$. The inequality $-2x_1 + 3x_2 \leq 3$ is a cutting plane for P as it separates only fractional points from P. The boundary of the half-space $(-2x_1+3x_2 \leq 4-\varepsilon)$ has been pushed towards P until an integer point is contained in it.

By Lemma 2.5, every (supporting) valid inequality $ax \leq a_0$ for $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ can be obtained as a positive linear combination of the rows of the system $Ax \leq b$. If $A \in \mathbb{R}^{m\times n}$, then the Gomory-Chvátal closure of P can be equivalently obtained as

$$P' = \bigcap_{\lambda \in [0,1)^m,\ \lambda A \in \mathbb{Z}^n} \Big((\lambda A)x \leq \lfloor \lambda b \rfloor \Big) \ . \tag{2.8}$$

A useful property of the Gomory-Chvátal closure operation is that it commutes with unimodular transformations.

Lemma 2.17 *Let P be a rational polyhedron and let $u : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a unimodular transformation. Then*

$$\big(u(P)\big)' = u(P') \ .$$

2.7.2 The Gomory-Chvátal Procedure

The elementary closure operation – that is, the application of all Gomory-Chvátal cuts to a polyhedron P – can be iterated so that a sequence of relaxations of the integer hull P_I of P is obtained. Defining $P^{(0)} := P$ and $P^{(k+1)} := (P^{(k)})'$, for $k \geq 0$, we have $P \supseteq P' \supseteq P^{(2)} \supseteq \ldots \supseteq P_I$. This iterative method is also called the *Gomory-Chvátal procedure.* Chvátal (1973) showed that for every rational polytope, there is a natural number t such that $P^{(t)} = P_I$. In particular, the above sequence is a sequence of successively tighter relaxations. Schrijver (1980) extended this result to unbounded polyhedra (and non-rational polytopes).

Theorem 2.18 *For each rational polyhedron P, there exists a natural number t such that $P^{(t)} = P_I$.*

The smallest number t with the property that $P^{(t)} = P_I$ is called the *Chvátal rank* of the polyhedron. Even though the number is finite for every rational polyhedron, it can be arbitrary large, even in dimension two (see Figure 2-2).

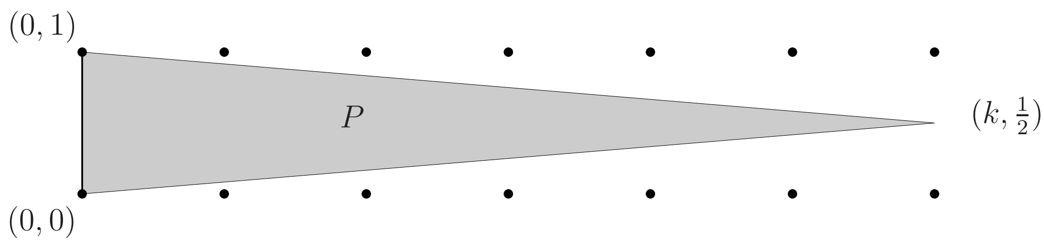

Figure 2-2: (Chvátal 1973) The polytope $P_k = \text{conv}\{(0,0);\ (0,1);\ (k, \frac{1}{2})\}$ has the property that $P_{(k-1)} \subseteq P'_k$. Because $P_I = \text{conv}\{(0,0);\ (0,1)\}$, the Chvátal rank of P_k is at least k. In particular, the rank of P_k is exponential in the encoding length of P_k, which is $O(\log(k))$.

In contrast to the above example, Bockmayr et al. (1999) proved that the Chvátal rank is polynomially bounded in the dimension n if the polytope is contained in the unit cube $[0,1]^n$. Eisenbrand and Schulz (2003) improved the upper bound:

Theorem 2.19 *If $P \subseteq [0,1]^n$, then the Chvátal rank of P is bounded by a function in $O(n^2 \log n)$.*

Chapter 3

The Complexity of Gomory-Chvátal Cuts with Coefficients 0 or 1

3.1 Introduction

In order to solve a linear programming problem with the ellipsoid method it is not necessary to rely on explicit inequality description of the underlying polyhedron. In fact, it suffices to be able to decide whether a given point is a solution of the problem and, if not, to find a violated inequality (see, e.g., Grötschel, Lovász, and Schrijver 1988, Karp and Papadimitriou 1980). In particular, if one can solve the separation problem for a polyhedron P in polynomial time, one can optimize any linear function over P in polynomial time. This observation is invaluable for the solution of combinatorial optimization problems with linear programming techniques as the polyhedra associated with these problems generally have too many facets to be listed explicitly.

Example 3.1 ***(Matching problem)*** *Given an undirected graph $G = (V, E)$ with vertex set V and edge set E, a* matching *$M \subseteq E$ in G is a set of pairwise non-adjacent edges; that is, no two edges share a common vertex. The* matching polytope *is defined as the convex hull of all incidence vectors associated with matchings in G and, hence, a* *subset of* $\mathbb{R}^{|E|}$*. A standard relaxation* $Q(G)$ *of the matching polytope is given by*

$$x_e \geq 0 \qquad \text{for all } e \in E \tag{3.1}$$

$$\sum_{e \in \delta(v)} x_e \leq 1 \qquad \text{for all } v \in V \ . \tag{3.2}$$

Here, $\delta(v)$ *denotes the set of edges incident to the node* v*. Clearly, every integral solution to the above system of constraints corresponds to a matching in* G*. However, in general,* $Q(G)$ *is not an integral polytope, that is,* $Q(G) \neq Q_I(G)$*. In order to obtain a linear description of the matching polytope, additional inequalities that are satisfied by all solutions associated with matchings of the graph have to be added to* $Q(G)$*. In particular, any matching in* G *has the property that for any odd subset* $U \subseteq V$ *of the nodes, it has no more than* $(|U|-1)/2$ *edges with both endpoints in* U*. If* $E(U)$ *denotes the set of edges with both endpoints in* U*, the constraints*

$$\textstyle\sum_{e \in E(U)} x_e \quad \leq \quad \frac{|U|-1}{2} \qquad \text{for all } U \subseteq V,\ |U| \text{ odd} \tag{3.3}$$

are valid for $Q_I(G)$*. In fact, Edmonds (1965) showed that the matching polytope is fully described by the constraints* (3.1), (3.2), *and* (3.3).

The matching polytope has an exponential number of facets. Yet, one can find a maximum matching in polynomial time. This is because the separation problem for *odd-set inequalities* (3.3) can be solved efficiently (Padberg and Rao 1982). (Note that one can check in polynomial time whether a given point satisfies the inequalities (3.1) and (3.2).) Another interesting observation is that the inequalities (3.3) are Gomory-Chvátal cuts for $Q(G)$. They correspond to the valid inequalities obtained by summing up the inequalities (3.2) for all nodes $v \in U$, adding the inequalities $-x_e \leq 0$ for every edge e with exactly one endpoint in U, and dividing the result by 2. Hence, the Chvátal rank of the relaxation $Q(G)$ is exactly 1.

A remarkable proportion of research in the integer programming community has been devoted to finding efficient separation procedures. Typically, one would first identify valid inequalities or facets of the convex hull of the feasible solutions to an integer programming problem and then derive efficient separation algorithms or heuristics for these inequalities.

As a result, many of these separation procedures refer to problem-specific cuts, that is, cutting planes that are derived combinatorially and require knowledge about the underlying structure of a problem. Impressive results have been obtained in this way for many important combinatorial optimization problems, especially in combination with branch-and-cut algorithms (see, e.g., Padberg and Rinaldi 1991).

While the general sentiment in the 1990's was that general cutting planes, such as Gomory-Chvátal cuts or lift-and-project cuts, were practically not useful, this perception started to change a few years later, when these cuts were successfully implemented by Balas, Ceria, and Cornuéjols (1993) (see also Balas et al. 1996 and Balas et al. 1996). As a consequence, this raised interest in the study of theoretical aspects of general cutting planes and, in particular, the separation and membership problems associated with certain subclasses of general cutting planes.

Definition 3.2 *The membership problem for the elementary closure of a family of cuts is:*

> *Given a rational polyhedron $P \subseteq \mathbb{R}^n$ and a rational point $\bar{x} \in P$, decide whether $\bar{x}$ does not belong to the elementary closure of P for this family of cuts.*

Regarding the family of Gomory-Chvátal cuts, Caprara and Fischetti (1996a) showed that the membership problem for the so-called $\left\{0, \frac{1}{2}\right\}$-closure is NP-complete. A $\left\{0, \frac{1}{2}\right\}$-cut for a polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ is a Gomory-Chvátal cut of the form $(\lambda A)x \leq \lfloor \lambda b \rfloor$, where λA is integral and each component of λ is either 0 or $\frac{1}{2}$. Eisenbrand (1999) proved the same hardness result for the membership problem of the Gomory-Chvátal closure. In particular, he observed that for the polyhedron constructed in the reduction of Caprara and Fischetti (1996a), the Gomory-Chvátal closure and the $\left\{0, \frac{1}{2}\right\}$-closure are identical.

In this chapter, we study the membership problem for the subclass of Gomory-Chvátal cuts $ax \leq \lfloor a_0 \rfloor$ for which all components of the normal vector a are either 0 or 1. Many combinatorially derived cutting planes are in fact Gomory-Chvátal cuts with this property, as for example the odd-set inequalities for the matching problem or the *odd-cycle inequalities* for the *stable set* problem. We show that the problem of deciding whether for a given point $\bar{x} \in P$ there exists a Gomory-Chvátal cut $ax \leq \lfloor a_0 \rfloor$ with $a \in \{0,1\}^n$ and such that $a\bar{x} > a_0$ is NP-complete.

3.2 Membership and Separation

The membership problem associated with the subclass of Gomory-Chvátal cuts with coefficients 0 or 1 is:

> *Given a rational polyhedron $P \subseteq \mathbb{R}^n$ and a rational point $\bar{x} \in P$, decide whether there exists a Gomory-Chvátal cut $ax \leq \lfloor a_0 \rfloor$ such that $a \in \{0,1\}^n$ and $a\bar{x} > \lfloor a_0 \rfloor$.*

In the following, we show that the membership problem for the elementary closure associated with the set of Gomory-Chvátal cuts with coefficients 0 or 1 is NP-complete. The proof is inspired by a reduction of Caprara and Fischetti (1996b).

Theorem 3.3 *The membership problem for the family of Gomory-Chvátal cuts with coefficients* 0 *or* 1 *is NP-complete.*

Proof. We describe a reduction from the *maximum cut* problem: Given an undirected graph $G = (V, E)$ and a positive integer K, decide whether there exists a subset $U \subseteq V$ such that the number of edges with exactly one endpoint in U is greater than K.

Consider an arbitrary instance of the maximum cut problem and define $n := |V|$ and $m := |E|$. Let $E = \{e_1, \ldots, e_m\}$ and assume that $e_j = \{u_j, v_j\}$ for $j = 1, \ldots, m$. For any subset $U \subseteq V$, we denote by $E(U)$ the set of edges with both endpoints in U and by $\delta(U)$ the set of edges with exactly one endpoint in U. We will construct a rational polyhedron $P \subseteq \mathbb{R}^n$ and a rational point $\bar{x} \in P$ such that there exists a Gomory-Chvátal cut with coefficients 0 or 1 separating $\bar{x}$ if and only if there exists a set $U \subseteq V$ such that $|\delta(U)| > K$.

We can assume w.l.o.g. that $K < m$, since the maximum cut problem can be solved easily for $K = m$. Let $M \in \{0,1\}^{n \times m}$ denote the incidence matrix of G, that is, $M_{ij} = 1$ if $\{i, j\} \in E$ and $M_{ij} = 0$, otherwise. Furthermore, for $i = 1, \ldots, n$, let $d_i := |\delta(i)|$ denote the degree of node i in G. We define $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, where $A \in \mathbb{Z}^{(m+n+1)\times(2m+n+1)}$ and $b \in \mathbb{Z}^{m+n+1}$ are specified in Figure 3-1. Furthermore, let

$$\bar{x} := \Big(\underbrace{\varepsilon \,,\ \ldots\ ,\ \varepsilon}_{m \text{ times}} \,,\ -\frac{\varepsilon d_1}{2} \,,\ \ldots\ ,\ -\frac{\varepsilon d_n}{2} \,,\ -\frac{(1+\varepsilon m)}{2} \,,\ \underbrace{0 \,,\ \ldots\ ,\ 0}_{m \text{ times}} \Big) \ ,$$

where $\varepsilon := 1/(m-K)$. We have

$$\begin{aligned} A\bar{x} &= \left(\varepsilon d_1 - 2\frac{\varepsilon d_1}{2}, \ldots, \varepsilon d_n - 2\frac{\varepsilon d_n}{2}, \varepsilon m - 2\frac{(1+\varepsilon m)}{2}, -\varepsilon, \ldots, -\varepsilon\right) \\ &= (0, \ldots, 0, -1, -\varepsilon, \ldots, -\varepsilon) \ , \end{aligned}$$

that is, $\bar{x} \in P$. Note that the sizes of A, b, and $\bar{x}$ are polynomially bounded in n and m.

$$\begin{array}{cc}
\begin{array}{c} n+1 \\[6em] m \end{array} &
Ax = \left(\begin{array}{c|c|c}
M & \begin{matrix} 2 & & \\ & \ddots & \\ & & 2 \end{matrix} & 0 \\ \cline{1-1}
1 \ \cdots \ 1 & & \\ \hline
\begin{matrix} -1 & & \\ & \ddots & \\ & & -1 \end{matrix} & 0 & \begin{matrix} 2 & & \\ & \ddots & \\ & & 2 \end{matrix}
\end{array}\right)
\left(\begin{array}{c} \\ \\ x \\ \\ \\ \end{array}\right)
\le
\left(\begin{array}{c} 0 \\ \hline -1 \\ \hline 0 \end{array}\right) = b \\
& \begin{array}{ccc} \quad m \quad & \quad n+1 \quad & \quad m \quad \end{array}
\end{array}$$

Figure 3-1: The constraint matrix A and right-hand side vector b that specify the constructed polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \le b\}$ in the NP-completeness proof.

In the following, we show that there exists a Gomory-Chvátal cut with coefficients 0 or 1 separating $\bar{x}$ if and only if there is a cut in the graph G of size greater than K. First, suppose there exists $U \subseteq V$ such that $|\delta(U)| > K$. We will derive a valid inequality for P as nonnegative linear combination of the rows of the system $Ax \le b$ such that the corresponding Gomory-Chvátal cut has all coefficients 0 or 1 and is violated by $\bar{x}$. Let $\lambda_1, \ldots, \lambda_{n+1}$ denote the multipliers associated with the first $n+1$ rows of $Ax \leq b$ and let $\mu_1, \ldots, \mu_m$ denote the multipliers corresponding to the last m rows. We define for $i = 1, \ldots, n$,

$$\lambda_i \quad := \quad \begin{cases} \frac{1}{2}, & \text{if } i \in U \\ 0, & \text{otherwise} \end{cases}$$

and set $\lambda_{n+1} := \frac{1}{2}$. Furthermore, we define

$$\mu_j := \begin{cases} \frac{1}{2}, & \text{if } e_j \in E \setminus \delta(U) \\ 0, & \text{otherwise} \end{cases} .$$

Now let $ax \leq a_0$ denote the valid inequality for P that is associated with these multipliers. We will show that each component of a is either 0 or 1. First, consider a component a_j corresponding to an edge of G, that is, $1 \leq j \leq m$. If $e_j = \{u_j, v_j\} \in \delta(U)$, then $|\{u_j, v_j\} \cap U| = 1$, implying $a_j = \frac{1}{2} + \frac{1}{2} = 1$. If $e_j = \{u_j, v_j\} \notin \delta(U)$, then either $|\{u_j, v_j\} \cap U| = 2$ or $|\{u_j, v_j\} \cap U| = 0$. In the former case, $a_j = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} - \frac{1}{2} = 1$, and in the latter case, $a_j = \frac{1}{2} - \frac{1}{2} = 0$. For $i = 1, \ldots, n+1$, we have $a_{m+i} = 2\lambda_i \in \{0, 1\}$, and for $j = 1, \ldots, m$, we get $a_{m+n+1+j} = 2\mu_j \in \{0, 1\}$. Hence, $a \in \{0, 1\}^{2m+n+1}$. The right-hand side of the valid inequality is $a_0 = -\lambda_{n+1} = -\frac{1}{2}$. Consequently, all coefficients of the Gomory-Chvátal cut $ax \leq -1$ are in $\{0, 1\}$. Moreover,

$$\begin{aligned}
a\bar{x} &= \sum_{j=1}^{m} a_j \varepsilon - \sum_{i=1}^{n} a_{m+i} \frac{\varepsilon d_i}{2} - a_{m+n+1} \frac{(1+\varepsilon m)}{2} \\
&= \sum_{j \in \delta(U)} \varepsilon + \sum_{j \in E(U)} \varepsilon - \sum_{i \in U} \frac{\varepsilon d_i}{2} - \frac{(1+\varepsilon m)}{2} \\
&= -\frac{1}{2} + \varepsilon \left(|\delta(U)| + |E(U)| - \sum_{i \in U} \frac{d_i}{2} - \frac{m}{2} \right) \\
&= -\frac{1}{2} + \frac{\varepsilon}{2} \big(|\delta(U)| - m \big) \\
&> -\frac{1}{2} + \frac{\varepsilon}{2} \big(K - m \big) = -1 \ ,
\end{aligned}$$

that is, $\bar{x}$ violates the cut.

For the other direction, suppose there exists a Gomory-Chvátal cut $ax \leq \lfloor a_0 \rfloor$ such that every component of a is either 0 or 1 and such that $a\bar{x} > \lfloor a_0 \rfloor$. Since $a_{m+n+1+j} = 2\mu_j \in \{0,1\}$, we must have $\mu_j \in \{0, \frac{1}{2}\}$, for $j = 1, \ldots, m$. Similarly, since $a_{m+i} = 2\lambda_i \in \{0,1\}$, we get $\lambda_i \in \{0, \frac{1}{2}\}$, for $i = 1, \ldots, n+1$. Furthermore, $\lambda_{n+1} \in \{0, \frac{1}{2}\}$ together with the fact that $\lfloor a_0 \rfloor = \lfloor -\lambda_{n+1} \rfloor \neq a_0$ implies that $\lambda_{n+1} = \frac{1}{2}$. Finally, it must hold for $j = 1, \ldots, m$ that

$$\mu_j = \frac{1}{2} \quad \text{if and only if} \quad \sum_{i=1}^{n} \lambda_i M_{ij} \in \{0,1\} \ .$$

This is because for $e_j = \{u_j, v_j\} \in E$, we have $a_j = \lambda_{u_j} + \lambda_{v_j} + \frac{1}{2} - \mu_j$. Now let us define

$$U := \left\{ i \in V \,\middle|\, \lambda_i = \frac{1}{2} \right\} \ .$$

Then it holds that

$$\mu_j = \frac{1}{2} \quad \text{if and only if} \quad e_j \in E \setminus \delta(U) \ ,$$

and therefore

$$\sum_{j=1}^{m} \mu_j = \frac{1}{2} |E \setminus \delta(U)| \ .$$

Finally, $a\bar{x} > -1$ implies that

$$\begin{aligned}
a\bar{x} &= \sum_{j=1}^{m} a_j \varepsilon - \sum_{i=1}^{n} a_{m+i} \frac{\varepsilon d_i}{2} - a_{m+n+1} \frac{(1+\varepsilon m)}{2} \\
&= \sum_{j=1}^{m} (\lambda_{u_j} + \lambda_{v_j} + \frac{1}{2} - \mu_j)\varepsilon - \sum_{i \in U} \frac{\varepsilon d_i}{2} - \frac{(1+\varepsilon m)}{2} \\
&= \varepsilon\big(|E(U)| + \frac{1}{2}|\delta(U)|\big) - \frac{\varepsilon}{2}|E \setminus \delta(U)| - \sum_{i \in U} \frac{\varepsilon d_i}{2} - \frac{1}{2} \\
&= -\frac{\varepsilon}{2}|E \setminus \delta(U)| - \frac{1}{2} > -1 \ ,
\end{aligned}$$

and therefore $|E \setminus \delta(U)| < 1/\varepsilon = m - K$. We obtain $|\delta(U)| > K$. $\square$

As the separation problem associated with a family of cutting planes is at least as hard as the membership problem, we obtain the following corollary.

Corollary 3.4 *Optimizing a linear function over the elementary closure associated with the family of Gomory-Chvátal cuts with coefficients either* 0 *or* 1 *cannot be done in polynomial time, unless* $P \neq NP$.

Chapter 4

The Gomory-Chvátal Closure of Non-rational Polytopes

4.1 Introduction

Gomory-Chvátal cutting planes first appeared in Gomory's cutting plane method (Gomory 1958), which provided a general framework for solving integer programming problems. Thus, they were originally introduced for polyhedra that can be described by rational data. Chvátal (1973) studied Gomory-Chvátal cuts more formally. In particular, he showed that the sequence of successively tighter relaxations arising from repeated applications of all Gomory-Chvátal cuts to a rational polytope P yields the integer hull P_I after finitely many steps. Schrijver (1980) generalized the result to unbounded rational polyhedra and non-rational polytopes.

Formally, the Gomory-Chvátal closure P' of a polyhedron P is the intersection of all half-spaces defined by the Gomory-Chvátal cuts of P. As their number is infinite, it is not clear from the definition whether P' is also polyhedron. Put differently, does a finite number of inequalities suffice to describe P'? Schrijver (1980) showed that, for a rational polyhedron, the Gomory-Chvátal cuts corresponding to a totally dual integral system of linear inequalities describing P specify its closure P' fully. For every rational polyhedron such a system is guaranteed to exist. Equivalently, the elementary closure is obtained by taking all Gomory-Chvátal cuts that are associated with normal vectors in the Hilbert bases of the basic feasible cones of P. Since the number of basic feasible cones of a polyhedron is finite and since every rational cone has a finite Hilbert basis, the number of undominated Gomory-Chvátal cuts is finite.

For polyhedra that cannot be described by rational data the situation is different. It is well-known that the integer hull of an unbounded non-rational polyhedron may not be a polyhedron (see, e.g., Halfin 1972). In fact, the integer hull may not be a closed set. Consider for example the non-rational polyhedral cone $P = \{x \in \mathbb{R}^2_+ \mid x_2 \leq \sqrt{2}x_1\}$. The half-line $\{x \in \mathbb{R}^2_+ \mid x_2 = \sqrt{2}x_1\}$ belongs to the topological closure of P_I and, therefore, no rational polyhedral cone can contain the same set of integer points as P. As a result, P_I is not a polyhedron. Regarding the Gomory-Chvátal closure of this non-rational polyhedral cone, it is not difficult to conclude that every Gomory-Chvátal cut for P is a valid inequality for P itself. In particular, we can approximate the non-rational facet-defining inequality $x_2 \leq \sqrt{2}x_1$ for P with arbitrary precision by a Gomory-Chvátal cut. This implies that P' cannot be described by a finite set of inequalities and, therefore, P' is not a rational polyhedron either. On the other hand, in the case of a non-rational polytope, P_I is the convex hull of a finite set of points and, therefore, a polytope. Yet, there is no notion of total dual integrality for non-rational systems of linear inequalities. Consequently, Schrijver (1980) writes in the discussion section of his paper, in which he proves the polyhedrality of the closure for rational polyhedra:

> *"We do not know whether the analogue ... is true in real spaces. We were able to show only that if P is a bounded polyhedron in real space, and P' has empty intersection with the boundary of P, then P' is a (rational) polyhedron."*

Since then, the question of whether the Gomory-Chvátal closure of a non-rational polytope is also a polytope has remained unanswered. In this part of the thesis, we resolve this open problem and prove that the Gomory-Chvátal closure of a non-rational polytope is a rational polytope. That is, a finite number of Gomory-Chvátal cuts suffice to describe the elementary closure. Apart from answering Schrijver's question, our result also extends a recent line of research concerning the polyhedrality of the Gomory-Chvátal closure of general convex sets. As Gomory-Chvátal cutting planes have been effective from a computational point of view (see, e.g., Balas et al. 1996, Bonami et al. 2008, and Fischetti and Lodi 2007), they have also been applied to general convex integer programs, that is, discrete optimization problems for which the continuous relaxation is a convex set, but not necessarily a linear program (see, e.g., Çezik and Iyengar 2005). Dey and Vielma (2010) showed that the Gomory-Chvátal closure of a full-dimensional ellipsoid described by rational data is a polytope. Shortly thereafter, Dadush, Dey, and Vielma (2010) extended this result to strictly convex bodies and to the intersection of a strictly convex body with a rational polyhedron. Since the proof of Schrijver (1980) for rational polyhedra strongly relies on polyhedral properties, Dey and Vielma and Dadush, Dey, and Vielma had to develop a new proof technique, which can be summarized as follows: First, it is shown that there exists a finite set of Gomory-Chvátal cuts that separate every non-integral point on the boundary of the strictly convex body. Second, one proves that, if there is a finite set of Gomory-Chvátal cuts for which the implied polytope is a subset of the body and for which the intersection of this polytope with the boundary of the body is contained in $\mathbb{Z}^n$, only a finite number of additional inequalities are needed to describe the closure of the body. The second step is strongly informed by Schrijver's observation that the boundary of the convex set is crucial for building the Gomory-Chvátal closure (see the quote above).

Our general proof strategy for showing the polyhedrality of the Gomory-Chvátal closure of a non-rational polytope is inspired by the work of Dadush et al. (2010). Yet, the key argument is very different and new, since Dadush et al.'s proof relies on properties of strictly convex bodies that do not extend to polytopes. More precisely, strictly convex bodies do not have any higher-dimensional "flat faces", and therein lies the main difficulty in establishing the polyhedrality of the elementary closure for non-rational polytopes. Our proof is geometrically motivated and applies ideas from convex analysis, polyhedral theory, and the geometry of numbers.

Dadush, Dey, and Vielma (2011) proved, at the same time and independently, that the Gomory-Chvátal closure of any compact convex set is a rational polytope.

4.2 Outline

The remainder of this chapter consists of two main sections. In Section 4.3, we first review the techniques used by Dadush et al. (2010) (Section 4.3.1) and then give an informal description of our proof (Section 4.3.2). The latter Section 4.3.2 is intended to introduce the main ideas of our approach, without going into technical details. Section 4.4 contains the actual proof, which is divided into four steps. We start by presenting some preliminary results in Section 4.4.1 and then prove each of the four steps separately in Sections 4.4.2 to 4.4.5.

4.3 General Proof Idea

Schrijver's proof of the polyhedrality of the Gomory-Chvátal closure for rational polyhedra uses totally dual integral systems, which do not exist for non-rational inequality systems. On the other hand, the proof technique of Dadush et al. (2010) for strictly convex bodies depends crucially on the fact that these sets do not have any higher-dimensional flat faces. Therefore, new ideas have to be developed to show that the Gomory-Chvátal closure of a non-rational polytope is a (rational) polytope. However, our general proof structure is inspired by the work of Dadush et al. (2010). Therefore, we outline briefly the two main steps of their argument for strictly convex bodies.

For this section and the sections to follow, some basic notation is required. Let $K \subseteq \mathbb{R}^n$ be a closed and convex set. For any vector $a \in \mathbb{R}^n$, we will denote by a_K the minimal right-hand side such that $ax \leq a_K$ is a valid inequality for K, that is, $a_K := \max\{ax \mid x \in K\}$. For every $S \subseteq \mathbb{Z}^n$, we define

$$C_S(K) := \bigcap_{a \in S} \big(ax \leq \lfloor a_K \rfloor\big) \ ,$$

that is, $C_S(K)$ is the intersection of all half-spaces defined by the Gomory-Chvátal cuts with normal vectors in S. In particular, the Gomory-Chvátal closure K' is obtained for $S = \mathbb{Z}^n$.

4.3.1 Outline of the Proof for Strictly Convex Bodies

The general idea of Dadush, Dey, and Vielma (2010) for showing that the Gomory-Chvátal closure of a strictly convex body K can be described by a finite set of Gomory-Chvátal cuts is the following: In the first step of their proof, they construct a finite set $S \subseteq \mathbb{Z}^n$, such that

$$(i) \quad C_S(K) \subseteq K \ , \tag{K1}$$

$$(ii) \quad C_S(K) \cap \mathrm{bd}(K) \subseteq \mathbb{Z}^n \ . \tag{K2}$$

In other words, $C_S(K)$ is a polytope that is contained in the original body and every point that it shares with the boundary of K is an integer point and, thus, is also contained in its closure K'. Consequently, any additional Gomory-Chvátal cut that is necessary in a description of K' has to separate a vertex of the polytope $C_S(K)$ that lies in the interior of K. In the second step of the proof, they argue that only a finite number of such cuts can exist and, hence, S needs to be augmented by, at most, a finite set of vectors.

The main difficulty of the proof is the argument for the existence of a finite set S satisfying properties (K1) and (K2). It can be roughly explained as follows: First, one shows that for every fractional point u on the boundary of K, there exists a Gomory-Chvátal cut that separates the point together with an open neighborhood around it. Such a cut can be constructed via a Diophantine approximation of some vector a in the normal cone of K at u. In this part of the proof Dadush et al. (2010) use a crucial property of strictly convex bodies: the sequence of points on the boundary of K that maximize the approximations of a converges to u. Subsequently, in the case that there are no integral boundary points, one can cover the boundary by a, possibly, infinite number of these open sets. Using a compactness argument, there is a finite sub-cover that yields a finite set of Gomory-Chvátal cuts separating the boundary of K. If the boundary contains integral points, one can construct for each of them a finite set of cuts that remove all points in an open neighborhood around the point. After removing these open neighborhoods from the boundary, a finite union of compact sets remains. Therefore, applying the compactness argument to each of these sets yields the existence of a finite set S of integer vectors with the desired properties.

The second observation, concerning the finite number of additional cuts, follows directly from the fact that for a vector $a \in \mathbb{Z}^n$, the maximum possible distance between the boundaries of the half-spaces $(ax \leq a_K)$ and $(ax \leq \lfloor a_K \rfloor)$, regardless of the right-hand side a_K, decreases with the norm of a (see Figure 4-2 for an explanation). Hence, for every vertex of $C_S(K)$ in the strict interior of K there is a threshold for the norm of the normal vector a, above which the associated Gomory-Chvátal cut cannot separate the

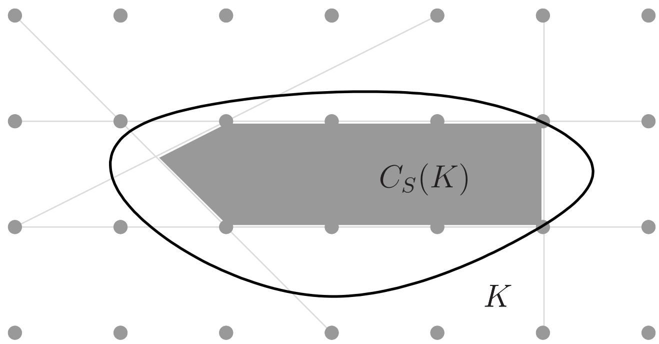

Figure 4-1: First part of the proof for strictly convex bodies: construction of a finite set S of integral vectors satisfying $C_S(K) \subseteq K$ and $C_S(K) \cap \mathrm{bd}(K) \subseteq \mathbb{Z}^n$.

vertex. The assumption of full-dimensionality of K is crucial to this part of the proof.

4.3.2 Outline of the Proof for Non-rational Polytopes

As Section 4.3.1 illustrates, the proof of Dadush et al. (2010) relies on two assumptions that prevent us from applying their technique to non-rational polytopes: *strict* convexity and full-dimensionality. Strict convexity is crucial for the first part of their proof, where they show the existence of a finite set $S \subseteq \mathbb{Z}^n$ that satisfies $C_S(K) \subseteq K$ and $C_S(K) \cap \mathrm{bd}(K) \subseteq \mathbb{Z}^n$. In particular, they demonstrate that every fractional point on the boundary of the strictly convex body is separated by a Gomory-Chvátal cut. Obviously, the same cannot be true for polytopes, since this would otherwise imply that the Gomory-Chvátal procedure separates fractional points in the relative interior of the facets of an integral polytope. On the other hand, if the Gomory-Chvátal closure P' of any *full-dimensional* polytope P is indeed a polytope, an intuitive modification of the requirements (K1) and (K2) is the existence of a finite set $S \subseteq \mathbb{Z}^n$ that satisfies

$$(i) \quad C_S(P) \subseteq P \ , \tag{P1a}$$

$$(ii) \quad C_S(P) \cap \mathrm{bd}(P) \subseteq P' \ . \tag{P2a}$$

Similar to the proof for strictly convex bodies, any additional undominated Gomory-Chvátal cut would have to separate a vertex of the polytope $C_S(P)$ in the strict interior

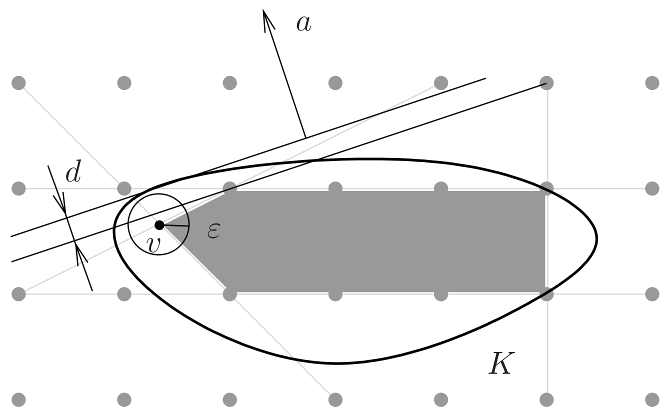

Figure 4-2: Second part of the proof for strictly convex bodies: If v is an interior point of the body K, one can find an ε-ball around it with $B(v, \varepsilon) \subseteq K$. For a Gomory-Chvátal cut $ax \leq \lfloor a_K \rfloor$ for K to separate v, the distance d between the hyperplanes $(ax = a_K)$ and $(ax = \lfloor a_K \rfloor)$ has to be at least ε. Since $\|a\|^{-1}$ is an upper bound on d, only normal vectors a with $\|a\| \leq \varepsilon^{-1}$ have to be considered.

of P and, as illustrated in Figure 4-2, the number of such cuts is finite. However, the full-dimensionality of P that is required in the above argument cannot generally be assumed. With regard to rational polytopes, non-full-dimensionality is not an obstacle, since one can find a unimodular transformation that maps P to a full-dimensional polytope in a lower-dimensional space (see Lemma 2.17). Yet, such a rational transformation does not exist for polytopes with non-rational affine hulls. In consequence, it becomes necessary to further modify the second property (P2a): one can only hope to find a finite set of Gomory-Chvátal cuts that separate the points in the *relative* boundary of the polytope that do not belong to its closure.

With these considerations in mind, our general strategy for proving that for any polytope a finite number of Gomory-Chvátal cuts is sufficient to describe the polytope's closure is the following: First, we show that one can find a finite set S of integral vectors such that

$$(i) \quad C_S(P) \subseteq P \ , \tag{P1b}$$

$$(ii) \quad C_S(P) \cap \operatorname{rbd}(P) \subseteq P' \ . \tag{P2b}$$

We then argue that, given the polytope $C_S(P)$, no more than a finite number of additional Gomory-Chvátal cuts are necessary to describe the closure P'. More precisely, we use the fact that any additional (undominated) Gomory-Chvátal cut for P must separate a vertex of $C_S(P)$ in the *relative* interior of P. While, in contrast to full-dimensional bodies, their number is generally infinite, we show that only a finite subset of the Gomory-Chvátal cuts with this property need to be considered. The overall strategy of our proof is summarized in Figure 4-3.

Part I: Show that one can construct a finite set $S \subseteq \mathbb{Z}^n$ such that

$$(i) \quad C_S(P) \subseteq P \ , \tag{P1}$$

$$(ii) \quad C_S(P) \cap \mathrm{rbd}(P) \subseteq P' \ . \tag{P2}$$

Part II: Show that, given a set S as in Part I, only a finite number of Gomory-Chvátal cuts have to be added to $C_S(P)$ to obtain the Gomory-Chvátal closure P'.

Figure 4-3: General strategy for proving the polyhedrality of the Gomory-Chvátal closure of a non-rational polytope.

The main challenge of this proof strategy lies in Part I and, more specifically, in showing the existence of a set S satisfying property (P1). This is due to the presence of higher-dimensional flat faces for polytopes, that is, higher-dimensional faces with non-rational affine hulls. These require the development of completely new arguments compared to the proof for strictly convex bodies. While the high-level idea of our proof can be described by the two main parts illustrated in Figure 4-3 above, several building blocks are necessary to implement this strategy. Our actual proof is divided into four steps:

Step 1: Show that there exists a finite set $S_1 \subseteq \mathbb{Z}^n$ such that $C_{S_1}(P) \subseteq P$.

Step 2: Show that for any face F of P, $F' = P' \cap F$. In particular, if $F = P \cap (ax = a_P)$, then for every Gomory-Chvátal cut for F there exists a Gomory-Chvátal cut for P that has the same impact on the maximal rational affine subspace of $(ax = a_P)$.

Step 3: Show that if there exists a finite set S satisfying (P1) and (P2), then P' is a rational polytope.

Step 4: Prove that P' is a rational polytope by induction on the dimension of $P \subseteq \mathbb{R}^n$.

In the remainder of this subsection, we describe the reasoning behind each step of the proof and outline some of the applied techniques.

Step 1: Constructing a subset of P from a finite number of Gomory-Chvátal cuts

As the first step of the proof, we show that one can construct a finite set of Gomory-Chvátal cuts that define a polytope that is contained in P. In particular, we show the property (P1) of Figure 4-3 for some finite set S of integral vectors. This step is the most challenging part of the proof.

Imagine that we can find a set $S \subseteq \mathbb{Z}^n$ with $C_S(P) \subseteq P$ for some full-dimensional polytope $P \subseteq \mathbb{R}^n$ for which a non-rational inequality $ax \leq a_P$ is facet-defining. As this inequality cannot be facet-defining for the *rational* polytope $C_S(P)$, there must exist a finite set of Gomory-Chvátal cuts that dominate $ax \leq a_P$. More formally, there must exist a subset $S_a \subseteq S$ such that $C_{S_a}(P) \subseteq (ax \leq a_P)$. If V_R denotes the maximal rational affine subspace V_R of $(ax = a_P)$, that is, the affine hull of all rational points in $(ax = a_P)$, then the Gomory-Chvátal cuts associated with the vectors in S_a have to separate every point in $(ax = a_P) \setminus V_R$. If this were not the case, $C_{S_a}(P)$ would have a non-rational face and, therefore, would not be a rational polyhedron. Regarding the points in the rational part of the boundary of the half-space $(ax \leq a_p)$, one can imagine that a slight "rotation of the boundary $(ax = a_P)$ around V_R" will result in a rational hyperplane that contains V_R and that corresponds to or is dominated by a Gomory-Chvátal cut for P.

Indeed, our strategy for the first step of the proof is to show that for each non-rational facet-defining inequality $ax \leq a_P$ for P there exists a finite set of integral vectors S_a that satisfies $C_{S_a}(P) \subseteq (ax \leq a_P)$. This fact is proven in a series of steps.

First, we establish the existence of a sequence of integral vectors satisfying a specific list of properties. These vectors give rise to Gomory-Chvátal cuts that separate all points in a non-rational facet $F = P \cap (ax = a_P)$ that are not contained in the maximal rational affine subspace V_R of $(ax = a_P)$. The number of Gomory-Chvátal cuts needed in our construction for separating the points in $(ax = a_P) \setminus V_R$ depends only on the dimension of V_R. If $\dim(V_R) = n - 2$, that is, the hyperplane $(ax = a_P)$ has a single "non-rational direction", then only two cuts are necessary. One can visualize these cuts to form a kind of "tent" in the half-space $(ax \leq a_P)$, with the ridge being V_R (see Figure 4-4 for an illustration). With each decrease in the dimension of V_R by 1, the number of necessary cuts is doubled. Hence, at most 2^{n-1} Gomory-Chvátal cuts are required to separate the non-rational parts of a non-rational facet of a polytope.

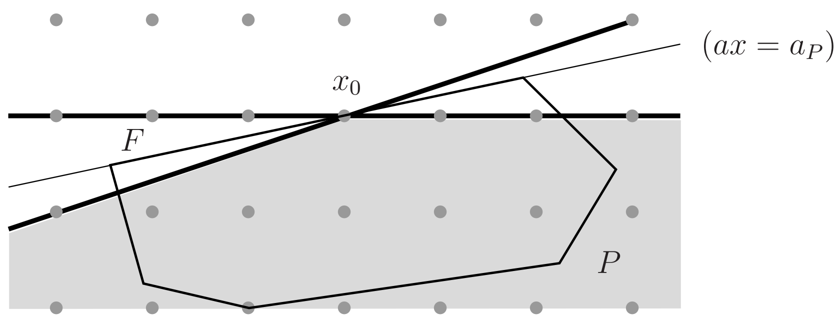

Figure 4-4: Construction of a finite set of Gomory-Chvátal cuts that dominate a non-rational facet-defining inequality $ax \leq a_P$. Here, the hyperplane $(ax = a_P)$ contains only one rational, in fact, one integral point x_0 and has, therefore, one non-rational direction. That is, $V_R = \{x_0\}$ and $\dim(V_R) = 0$. Two Gomory-Chvátal cuts separate every point in the hyperplane $(ax = a_P)$ that is not in V_R.

The proof of Step 1 uses many classic results from convex and polyhedral theory, as well as from number theory (e.g., Diophantine approximations, integral lattices, and reduced lattice bases).

Step 2: A homogeneity property: $F' = P' \cap F$

As the second step of the proof, we show a property of the Gomory-Chvátal closure that is well-known for rational polytopes (see, e.g., Schrijver 1986): If one applies the closure operator to a face of a polytope, the result is the same as if one intersected the closure of the polytope with the face. As it turns out, the same is true for non-rational polytopes. The proof for the rational case is based on the observation that any Gomory-Chvátal cut for a face $F = P \cap (ax = a_P)$ can be "rotated" so that it becomes a valid Gomory-Chvátal cut for P. In particular, the rotated cut has the same impact on the hyperplane $(ax = a_P)$ as the original cut for F. While the exact same property does not hold in the non-rational case, we show that there is a rotation of any cut for F that results in a Gomory-Chvátal cut for P, which has the same impact on the maximal rational affine subspace V_R of $(ax = a_P)$. As Step 1 of our proof implies that the non-rational parts of a face are separated in the first round of the Gomory-Chvátal procedure in any event, this property suffices to show that $F' = P' \cap F$.

The insights gained in this part of the proof will be useful for Step 4 of the proof, where we show the main result by induction on the dimension of the polytope. Knowing that the Gomory-Chvátal closure of a lower-dimensional facet F of P is a polytope, each of the finite number of cuts describing F' can be rotated in order to become a Gomory-Chvátal cut for P. We thereby establish the existence of a finite set $S_F \subseteq \mathbb{Z}^n$ with the property that $C_{S_F}(P) \cap F = F'$. Since the facets of P constitute the relative boundary of the polytope, the union of all these sets will give rise to a set $S \subseteq \mathbb{Z}^n$ that satisfies property (P2) in Figure 4-3.

Step 3: Finite augmentation property

In Step 3 of our proof, we show Part II of the general strategy outlined in Figure 4-3: Once a finite set $S \subseteq \mathbb{Z}^n$ of vectors is established with the property that $C_S(P) \subseteq P$ and $C_S(P) \cap \operatorname{rbd}(P) \subseteq P'$, not more than a finite number of Gomory-Chvátal cuts have to be added to obtain P'.

A similar result has been shown in Dadush, Dey, and Vielma (2010) for full-dimensional convex sets, that is, when the relative boundary of the set coincides with its boundary. In this case, it is possible to find an ε-ball around every vertex of $C_S(P)$ in the strict interior of P with the property that the ball is completely contained in P. Any additional, undominated Gomory-Chvátal cut must separate such a vertex and, hence, the boundary of the associated half-space containing P must be shifted by at least the radius of the ball. This is only possible up to a certain size of the normal vector, which implies the result (see also Figure 4-2).

Since a non-rational polytope P can be contained in some non-rational affine subspace and, hence, a unimodular transformation of P to a full-dimensional polytope in a lower-dimensional space does not exist, an extension of the result to non-full-dimensional polytopes is required. In that case, any additional Gomory-Chvátal cut that is needed to describe P' must separate a vertex of $C_S(P)$ in the *relative* interior of P. However, the ε-ball argument is not valid in this situation, as one can only find an ε-ball around the vertex whose intersection with the affine hull of P is contained in P. While in the lower-dimensional affine space a similar argument as above applies, yet an infinite number of Gomory-Chvátal cuts in $\mathbb{R}^n$ remain, which can separate such vertex. Therefore, further arguments are needed to prove the finite augmentation property for non-full-dimensional convex sets.

Step 4: Proof of the main result

As the final step of the proof, we establish the main result of this chapter, namely that the Gomory-Chvátal closure of any polytope can be described by a finite set of inequalities. The proof is by induction on the dimension of the polytope and uses the observations made in the steps above. Step 1 provides a finite set $S_1 \subseteq \mathbb{Z}^n$ satisfying $C_{S_1}(P) \subseteq P$. Applying the induction assumption to the facets of P and using the homogeneity property of Step 2, we augment S_1 for each facet F by a finite set $S_F \subseteq \mathbb{Z}^n$ such that the resulting set of integral vectors satisfies properties (P1) and (P2) of Part I in Figure 4-3. From that it follows with the finite augmentation property proven in Step 3 that P' is a polytope.

4.4 The Proof

In this section, we prove the main result of Chapter 4: the Gomory-Chvátal closure of any polytope can be described by a finite set of inequalities. We start in Section 4.4.1

by introducing a collection of preliminary results that are required for the main proof. Following these preparatory considerations, in Sections 4.4.2 to 4.4.5 we separately prove the four steps outlined in Section 4.3.2.

4.4.1 Preliminary Results

In this subsection, we derive a number of preliminary results regarding Diophantine approximations and bases of lattices that are utilized in the subsequent sections to show that the Gomory-Chvátal closure of any polytope is a rational polytope.

The first lemma relates the coefficients in the representation of a point in $\mathbb{R}^n$ as a linear combination of a set of basis vectors to the coefficients of the representation of the same point with respect to some corresponding orthogonal basis.

Lemma 4.1 *Let $R > 0$ and let $u_1, \ldots, u_k, w_1, \ldots, w_l$ be linearly independent vectors in $\mathbb{R}^n$ with $\|w_j\| = R$, for $j = 1, \ldots, l$. Furthermore, let us define the linear vector spaces $U_0 := \operatorname{span}(u_1, \ldots, u_k)$ and $U_j := \operatorname{span}(u_1, \ldots, u_k, w_1, \ldots, w_j)$, for $j = 1, \ldots, l$. Let $\tilde{w}_j$ denote the orthogonal projection of w_j onto $U_{j-1}^{\perp}$, for $j = 1, \ldots, l$. If there exists a constant $c > 0$ such that $\|\tilde{w}_j\| \geq cR$ for $j = 1, \ldots, l$, then there exists a constant c_1 only depending on l and c such that*

$$\begin{aligned} &\left\{ u + \sum_{j=1}^{l} \tilde{\lambda}_j \tilde{w}_j \,\middle|\, u \in U;\ -1 \leq \tilde{\lambda}_j \leq 1,\ j = 1, \ldots, l \right\} \\ \subseteq\ &\left\{ u + \sum_{j=1}^{l} \lambda_j w_j \,\middle|\, u \in U;\ -c_1 \leq \lambda_j \leq c_1,\ j = 1, \ldots, l \right\} . \end{aligned}$$

Proof. The proof of the lemma is by induction on l. For $j = 1, \ldots, l$, the orthogonal projection $\tilde{w}_j$ of w_j onto $U_{j-1}^{\perp}$ has a unique representation (see also Section 2.2.2):

$$\tilde{w}_j \;=\; w_j - \sum_{p=1}^{k} \alpha_{jp} u_p - \sum_{t=1}^{j-1} \alpha_{jt} \tilde{w}_t \ , \tag{4.1}$$

where $\alpha_{jp} \in \mathbb{R}$ for $p = 1, \ldots, k$, and

$$\alpha_{jt} = \frac{w_j \tilde{w}_t}{\|\tilde{w}_t\|^2}$$

for $t = 1, \ldots, j-1$. First, consider the case $l = 1$. Take an arbitrary $x = u + \tilde{\lambda}_1 \tilde{w}_1$, where $u \in U$ and $\tilde{\lambda}_1 \in [-1, 1]$. Then

$$x = u + \tilde{\lambda}_1 \tilde{w}_1 = u + \tilde{\lambda}_1 \left(w_1 - \sum_{p=1}^{k} \alpha_{1p} u_p \right) = \left(u - \tilde{\lambda}_1 \sum_{p=1}^{k} \alpha_{1p} u_p \right) + \tilde{\lambda}_1 w_1 \ ,$$

and $c_1 = 1$ satisfies the conditions of the lemma. Therefore, assume that the statement of the lemma is true for some $l \geq 1$ with constant $c_1 = c_1(l, c)$. Now take an $x = u + \sum_{j=1}^{l+1} \tilde{\lambda}_j \tilde{w}_j$, where $u \in U$ and $\tilde{\lambda}_j \in [-1, 1]$ for $j = 1, \ldots, l+1$. Using the induction assumption and (4.1), we get

$$\begin{aligned}
x &= u + \sum_{j=1}^{l} \tilde{\lambda}_j \tilde{w}_j + \tilde{\lambda}_{l+1} \tilde{w}_{l+1} \\
&= u' + \sum_{j=1}^{l} \lambda_j w_j + \tilde{\lambda}_{l+1} \left(w_{l+1} - \sum_{p=1}^{k} \alpha_{l+1p} u_p - \sum_{t=1}^{l} \frac{w_{l+1} \tilde{w}_t}{\|\tilde{w}_t\|^2} \tilde{w}_t \right) \\
&= u'' + \sum_{j=1}^{l} \lambda_j w_j + \tilde{\lambda}_{l+1} \left(w_{l+1} - \sum_{j=1}^{l} \frac{w_{l+1} \tilde{w}_j}{\|\tilde{w}_j\|^2} \tilde{w}_j \right) \ ,
\end{aligned}$$

for some $u', u'' \in U$ and numbers λ_j satisfying $|\lambda_j| \leq c_1(l, c)$, for $j = 1, \ldots, l$. Let us define

$$y := \sum_{j=1}^{l} \frac{w_{l+1} \tilde{w}_j}{\|\tilde{w}_j\|^2} \tilde{w}_j = \sum_{j=1}^{l} \nu_j \tilde{w}_j \ .$$

Then

$$|\nu_j| = \frac{|w_{l+1} \tilde{w}_j|}{\|\tilde{w}_j\|^2} \leq \frac{\|w_{l+1}\| \, \|\tilde{w}_j\|}{\|\tilde{w}_j\|^2} = \frac{\|w_{l+1}\|}{\|\tilde{w}_j\|} \leq \frac{R}{Rc} = \frac{1}{c} \ .$$

By applying the induction assumption a second time, we get

$$\begin{aligned} y &\in \left\{ u + \sum_{j=1}^{l} \nu_j \tilde{w}_j \,\middle|\, u \in U;\ -\frac{1}{c} \le \nu_j \le \frac{1}{c},\ j = 1, \dots, l \right\} \\ &= \frac{1}{c}\left\{ u + \sum_{j=1}^{l} \nu_j \tilde{w}_j \,\middle|\, u \in U;\ -1 \le \nu_j \le 1,\ j = 1, \dots, l \right\} \\ &\subseteq \frac{1}{c}\left\{ u + \sum_{j=1}^{l} \gamma_j w_j \,\middle|\, u \in U;\ -c_1 \le \gamma_j \le c_1,\ j = 1, \dots, l \right\} \\ &= \left\{ u + \sum_{j=1}^{l} \gamma_j w_j \,\middle|\, u \in U;\ -\frac{c_1}{c} \le \gamma_j \le \frac{c_1}{c},\ j = 1, \dots, l \right\} . \end{aligned}$$

In particular, there exists some $u''' \in U$ and numbers $\gamma_j \in [-c_1/c, c_1/c]$, for $j = 1, \dots, l$, such that

$$y = u''' + \sum_{j=1}^{l} \gamma_j w_j \ .$$

Hence, we obtain

$$\begin{aligned} x &= u'' + \sum_{j=1}^{l} \lambda_j w_j + \tilde{\lambda}_{l+1} \left(w_{l+1} - u''' - \sum_{j=1}^{l} \gamma_j w_j \right) \\ &= \hat{u} + \sum_{j=1}^{l} (\lambda_j - \tilde{\lambda}_{l+1}\gamma_j)\, w_j + \tilde{\lambda}_{l+1} w_{l+1} \ , \end{aligned}$$

where $\hat{u} \in U$ and

$$\left| \lambda_j - \tilde{\lambda}_{l+1}\gamma_j \right| \le |\lambda_j| + |\gamma_j| \le c_1(l, c) + \frac{c_1(l, c)}{c} \ .$$

Thus, $c_1(l + 1, c) := c_1(l, c)\,(1 + 1/c)$ is the desired constant for $l + 1$. □

Next, we review a famous result regarding simultaneous Diophantine approximations: a finite set of real numbers can be approximated by rational numbers with one common low denominator.

Theorem 4.2 ***(Dirichlet's theorem)*** *For $a \in \mathbb{R}^n$ and $0 < \varepsilon < 1$, there exist inte-*

gers $p_1, \ldots, p_n$ *and* $q > 0$ *such that for* $i = 1, \ldots, n$,

$$\left| a_i - \frac{p_i}{q} \right| < \frac{\varepsilon}{q} \ .$$

In the following lemma, we extend Dirichlet's Theorem to rational approximations that retain rational linear dependencies between the components of the vector $a \in \mathbb{R}^n$.

Lemma 4.3 *Let* $a \in \mathbb{R}^n$ *and let* $u_1, \ldots, u_k$*, where* $k \le n - 1$*, be linearly independent vectors in* $\mathbb{Z}^n$ *such that* $au_j = 0$ *for* $j = 1, \ldots, k$*. For any* $0 < \varepsilon < 1$*, there exists an integer vector* $p = (p_1, \ldots, p_n)$ *and an integer* $q > 0$ *such that* $pu_j = 0$ *for* $j = 1, \ldots, k$*, and such that for* $i = 1, \ldots, n$,

$$\left| a_i - \frac{p_i}{q} \right| < \frac{\varepsilon}{q} \ .$$

Proof. Let U denote the $k \times n$ matrix with rows $u_1, \ldots, u_k$, that is, $Ua = 0$. Since $\operatorname{rank}(U) = k$, there exists (after possibly reordering the indices) a rational $k \times (n-k)$ matrix $\tilde{U}$ such that the system of equalities $Ua = 0$ is equivalent to the system

$$\begin{bmatrix} a_{n-k+1} \\ \vdots \\ a_n \end{bmatrix} = \tilde{U} \begin{bmatrix} a_1 \\ \vdots \\ a_{n-k} \end{bmatrix} \ .$$

In particular, one can find a positive integer s and integers r_{ij}, for $n - k + 1 \le i \le n$ and $j = 1, \ldots, n-k$, such that for $i = n - k + 1, \ldots, n$,

$$a_i = \frac{1}{s} \sum_{j=1}^{n-k} r_{ij}\, a_j \ .$$

Let us define the constants

$$K_1 := \min \left\{ \frac{1}{s}, \frac{s}{(n-k) \max_{i,j} |r_{ij}|} \right\}$$

and $\varepsilon_1 := K_1\, \varepsilon$. Let $\tilde{p}_1, \ldots, \tilde{p}_{n-k}$ and $\tilde{q}$ be integers according to Theorem 4.2 that satisfy

$$\left| a_i - \frac{\tilde{p}_i}{\tilde{q}} \right| < \frac{\varepsilon_1}{\tilde{q}}$$

for $i = 1, \ldots, n-k$. We define

$$\begin{aligned} q &:= s\,\tilde{q} \\ p_i &:= s\,\tilde{p}_i && \text{for } i = 1, \ldots, n-k \\ p_i &:= \sum_{j=1}^{n-k} r_{ij}\,\tilde{p}_j && \text{for } i = n-k+1, \ldots, n \ . \end{aligned}$$

Note that

$$\begin{bmatrix} p_{n-k+1} \\ \vdots \\ p_n \end{bmatrix} = \tilde{U} \begin{bmatrix} p_1 \\ \vdots \\ p_{n-k} \end{bmatrix} \ ,$$

implying $pu_j = 0$ for $j = 1, \ldots, k$. Furthermore, for $i = 1, \ldots, n-k$, we have

$$\left| a_i - \frac{p_i}{q} \right| = \left| a_i - \frac{\tilde{p}_i}{\tilde{q}} \right| < \frac{\varepsilon_1}{q/s} \leq \frac{\varepsilon}{q} \ .$$

Then we obtain for $i = n-k+1, \ldots, n$,

$$\left| a_i - \frac{p_i}{q} \right| = \left| a_i - \frac{1}{s} \sum_{j=1}^{n-k} r_{ij}\, \frac{\tilde{p}_j}{\tilde{q}} \right| = \frac{1}{s} \left| \sum_{j=1}^{n-k} r_{ij}\, a_j - \sum_{j=1}^{n-k} r_{ij}\, \frac{\tilde{p}_j}{\tilde{q}} \right| \leq \frac{1}{s} \sum_{j=1}^{n-k} |r_{ij}| \left| a_j - \frac{\tilde{p}_j}{\tilde{q}} \right| \leq \frac{\varepsilon}{q} \ ,$$

and the lemma follows. □

From the last lemma, we obtain the following corollary.

Corollary 4.4 *Let $a \in \mathbb{R}^n$ and let $u_1, \ldots, u_k$, where $k \leq n-1$, be linearly independent vectors in $\mathbb{Z}^n$ such that $au_j = 0$ for $j = 1, \ldots, k$. Then there exists a sequence $\{a^i\}_{i \in \mathbb{N}} \subseteq \mathbb{Z}^n$ such that $a^i \perp u_j$ for $j = 1, \ldots, k$ and such that*

$$\left\| a^i \right\| \left\| \bar{a}^i - \bar{a} \right\| \longrightarrow 0 \ , \tag{4.2}$$

where $\bar{a} = a / \|a\|$ and $\bar{a}^i = a^i / \|a^i\|$.

Proof. According to Lemma 4.3, there exist integral sequences $\{p^i\}_{i \in \mathbb{N}} \subseteq \mathbb{Z}^n$

and $\{q^i\}_{i\in\mathbb{N}} \subseteq \mathbb{N}$ such that $p^i u_j = 0$ for $j = 1, \ldots, k$ and such that

$$\left\|q^i a - p^i\right\| \longrightarrow 0 \ .$$

Therefore,

$$q^i \|a\| - \left\|p^i\right\| \longrightarrow 0 \ .$$

Let us define $a^i := p^i$ for all $i \in \mathbb{N}$. Then clearly $a^i \perp u_j$ for $j = 1, \ldots, k$. Furthermore,

$$\begin{aligned}
\left\|a^i\right\| \left\|\bar{a} - \bar{a}^i\right\| &= \left\|\bar{a}\left\|a^i\right\| - a^i\right\| = \left\|a\frac{\|p^i\|}{\|a\|} - p^i\right\| = \left\|a\frac{\|p^i\|}{\|a\|} - q^i a + q^i a - p^i\right\| \\
&\leq \left\|a\left(\frac{\|p^i\|}{\|a\|} - q^i\right)\right\| + \left\|q^i a - p^i\right\| \\
&= \left|\left\|p^i\right\| - q^i \|a\|\right| + \left\|q^i a - p^i\right\| \longrightarrow 0 \ .
\end{aligned}$$

□

The next Theorem concerns lattices and their bases: we show that if $\Lambda \subseteq \mathbb{R}^n$ is an integral $(n-1)$-dimensional lattice and $u_1, \ldots, u_k$ a basis for some k-dimensional sublattice of Λ, then we can extend the sublattice basis by vectors $v_1, \ldots, v_{n-k-1}$ to a basis of Λ such that the vectors $v_1, \ldots, v_{n-k-1}$ satisfy properties that are characteristic for reduced lattice bases.

Theorem 4.5 *Let $(ax = 0)$ be a hyperplane that is spanned by n integral points in $\mathbb{R}^n$. Let $U_0 \subseteq (ax = 0)$ be a k-dimensional linear vector space spanned by integral vectors $u_1, \ldots, u_k$ and assume that $u_1, \ldots, u_k$ form a basis of the lattice defined by the integer points in U_0. Let $l := n - k - 1$. If $k \geq 1$, assume that for any $v \in \big((ax = 0) \cap \mathbb{Z}^n\big) \setminus U_0$,*

$$\|v\|^2 \geq \frac{1}{2}\left(\sum_{p=1}^{k} \|u_p\|\right)^2 \ . \tag{4.3}$$

Then there exist vectors $v_1, \ldots, v_l$ that satisfy the following properties:

(1) The vectors $u_1, \ldots, u_k, v_1, \ldots, v_l$ form a basis of the lattice $(ax = 0) \cap \mathbb{Z}^n$.

(2) If we define $U_j := \operatorname{span}(u_1, \ldots, u_k, v_1, \ldots, v_j)$ and if $\tilde{v}_j$ denotes the orthogonal projection of v_j onto $U_{j-1}^{\perp}$, for $j = 1, \ldots, l$, then there exists a positive constant c only depending on l such that for $j = 1, \ldots, l$,

$$\|\tilde{v}_j\| \geq c \|v_j\| \quad .$$

Proof. We prove the theorem by slightly modifying the lattice basis reduction algorithm of Lenstra, Lenstra, and Lovász (1982). More precisely, we construct integral vectors $v_1, \ldots, v_l$ that satisfy

$$v_j = \tilde{v}_j + \sum_{p=1}^{k} \alpha_{jp} u_p + \sum_{t=1}^{j-1} \alpha_{jt} \tilde{v}_t \tag{4.4}$$

with

$$|\alpha_{jp}| \leq \frac{1}{2} \qquad \text{for } 1 \leq j \leq l \text{ and } p = 1, \ldots, k \tag{4.5}$$

$$|\alpha_{jt}| \leq \frac{1}{2} \qquad \text{for } 1 \leq j \leq l \text{ and } t = 1, \ldots, j-1 \tag{4.6}$$

and

$$\|\tilde{v}_{j+1} + \alpha_{j+1,j} \tilde{v}_j\|^2 \geq \frac{3}{4} \|\tilde{v}_j\|^2 \qquad \text{for } j = 1, \ldots, l-1 \quad . \tag{4.7}$$

Note that condition (4.7) says that

$$\operatorname{dist}^2\bigl(v_{j+1}, U_{j-1}\bigr) \geq \frac{3}{4} \operatorname{dist}^2\bigl(v_j, U_{j-1}\bigr) \qquad \text{for } j = 1, \ldots, l-1 \quad .$$

Suppose that $v_1, \ldots, v_l$ are integral vectors such that $u_1, \ldots, u_k, v_1, \ldots, v_l$ form a basis of the lattice $(ax = 0) \cap \mathbb{Z}^n$ and such that (4.5), (4.6), and (4.7) hold. Then property (1) of the theorem is clearly satisfied. Now consider property (2). Condition (4.7) implies that

$$\|\tilde{v}_j\|^2 \leq \frac{4}{3} \|\tilde{v}_{j+1}\|^2 + \frac{4}{3} \alpha_{j+1,j}^2 \|\tilde{v}_j\|^2 \leq \frac{4}{3} \|\tilde{v}_{j+1}\|^2 + \frac{1}{3} \|\tilde{v}_j\|^2 \quad .$$

Therefore, for $j = 1, \dots, l-1$,

$$\|\tilde{v}_{j+1}\|^2 \geq \frac{1}{2}\,\|\tilde{v}_j\|^2 \quad ,$$

and consequently for $1 \leq i \leq j \leq l$,

$$\|\tilde{v}_j\|^2 \geq \frac{1}{2^{j-i}}\,\|\tilde{v}_i\|^2 \quad . \tag{4.8}$$

From representation (4.4) for $j = 1$, we obtain with assumption (4.3) and (4.5),

$$\begin{aligned} \|v_1\|^2 &= \left\|\tilde{v}_1 + \sum_{p=1}^{k} \alpha_{1p} u_p\right\|^2 = \|\tilde{v}_1\|^2 + \left\|\sum_{p=1}^{k} \alpha_{1p} u_p\right\|^2 \\ &\leq \|\tilde{v}_1\|^2 + \frac{1}{4}\left(\sum_{p=1}^{k} \|u_p\|\right)^2 \leq \|\tilde{v}_1\|^2 + \frac{1}{2}\,\|v_1\|^2 \quad , \end{aligned}$$

and hence,

$$\|\tilde{v}_1\|^2 \geq \frac{1}{2}\,\|v_1\|^2 \quad . \tag{4.9}$$

Then, using assumptions (4.3), (4.5), (4.6), (4.8), and (4.9), we get for $j = 2, \dots, l$,

$$\begin{aligned} \|v_j\|^2 &= \left\|\tilde{v}_j + \sum_{p=1}^{k} \alpha_{jp} u_p + \sum_{t=1}^{j-1} \alpha_{jt} \tilde{v}_t\right\|^2 \\ &= \|\tilde{v}_j\|^2 + \left\|\sum_{p=1}^{k} \alpha_{jp} u_p\right\|^2 + \sum_{t=1}^{j-1} \alpha_{jt}^2\,\|\tilde{v}_t\|^2 \\ &\leq \|\tilde{v}_j\|^2 + \frac{1}{4}\left(\sum_{p=1}^{k} \|u_p\|\right)^2 + \frac{1}{4}\sum_{t=1}^{j-1} \|\tilde{v}_t\|^2 \\ &\leq \|\tilde{v}_j\|^2 + \frac{1}{2}\,\|v_1\|^2 + \frac{1}{4}\sum_{t=1}^{j-1} 2^{j-t}\,\|\tilde{v}_j\|^2 \\ &\leq \|\tilde{v}_j\|^2 + \|\tilde{v}_1\|^2 + \frac{1}{4}\sum_{t=1}^{j-1} 2^{j-t}\,\|\tilde{v}_j\|^2 \\ &\leq \left(1 + 2^{j-1} + \frac{1}{4}\big(2^j - 2\big)\right)\|\tilde{v}_j\|^2 \quad . \end{aligned}$$

Consequently, we can find some constant c only depending on l such that $\|\tilde{v}_j\| \geq c\,\|v_j\|$ for $j = 1, \ldots, l$, and property (2) is satisfied.

Procedure 1: Given the current basis $u_1, \ldots, u_k, v_1, \ldots, v_l$, compute the orthogonal projections of $v_1, v_2, \ldots, v_l$ onto $U_0, U_1, \ldots, U_{l-1}$, respectively; that is, the representations according to (4.4). Then for $j = 1, \ldots, l$, do the following:

(i) For $p = 1, \ldots, k$: If $|\alpha_{jp}| > \frac{1}{2}$, replace v_j by $v_j - [\alpha_{jp}]u_p$, where $[\alpha_{jp}]$ denotes the closest integer to α_{jp}.

(ii) For $t = 1, \ldots, j-1$: If $|\alpha_{jt}| > \frac{1}{2}$, replace v_j by $v_j - [\alpha_{jt}]v_t$.

Procedure 2: If for some index $s \in \{1, \ldots, l\}$

$$\|\tilde{v}_{s+1} + \alpha_{s+1,s}\tilde{v}_s\|^2 < \frac{3}{4}\,\|\tilde{v}_s\|^2 \quad ,$$

then swap v_s and v_{s+1} and go to Procedure 1.

Figure 4-5: A modified lattice basis reduction algorithm.

In the remainder of the proof, we explain an algorithm that constructs vectors $v_1, \ldots, v_l$ that satisfy (4.5), (4.6), and (4.7) and that have the property that $u_1, \ldots, u_k, v_1, \ldots, v_l$ form a basis of $\Lambda = (ax = 0) \cap \mathbb{Z}^n$. First, observe that we can assume that we are given some integral vectors $v_1, \ldots, v_l$ such that $u_1, \ldots, u_k, v_1, \ldots, v_l$ is a basis of the lattice Λ. Now consider the algorithm shown in Figure 4-5. The algorithm will modify the current basis by repeated applications of two procedures. Clearly, if the algorithm terminates, then $v_1, \ldots, v_l$ satisfy (4.5), (4.6), and (4.7). We therefore need to show that each of the two procedures is finite and executed only a finite number of times. First, observe that Procedure 1 requires at most $\sum_{j=1}^{l}(p + j - 1)$ changes of vectors, that is, the procedure is finite. Regarding Procedure 2, let $\tilde{u}_1 := u_1$ and let $\tilde{u}_p$ denote the orthogonal projection of u_p onto $(\mathrm{span}(u_1, \ldots, u_{p-1}))^{\perp}$. Let us define the quantities

$$d_j := \left(\prod_{p=1}^{k} \|\tilde{u}_p\|^2\right)\left(\prod_{t=1}^{j} \|\tilde{v}_t\|^2\right)$$

and

$$D := \prod_{j=1}^{l} d_j \ .$$

Note that Procedure 1 does not change the orthogonal projections of the vectors $v_1, \ldots, v_l$, and in particular, Procedure 1 does not affect the number D. However, D changes when there is a swap in Procedure 2. More precisely, d_s is reduced by a factor smaller than $\frac{3}{4}$, whereas all other d_j are unchanged. Let B_j denote the matrix with columns $u_1, \ldots, u_k, v_1, \ldots, v_j$, for $j = 1, \ldots, l$. Note that by (2.2), for $j = 1, \ldots, l$,

$$d_j = \left(\prod_{p=1}^{k} \|\tilde{u}_p\|^2 \right) \left(\prod_{t=1}^{j} \|\tilde{v}_t\|^2 \right) = \left| \det(B_j^T B_j) \right| \geq 1 \ .$$

In other words, d_j is the square of the determinant of the lattice $\Lambda_j = U_j \cap \mathbb{Z}^n$ and bounded from below. It follows that there is a lower bound on D and, hence, an upper bound on the number of times that we execute Procedure 2.

□

4.4.2 Step 1

In this subsection, we complete the first and most difficult step of our proof outlined in Section 4.3.2 above: we show that for any polytope P, there exists a finite set of Gomory-Chvátal cuts that defines a subset of the polytope. In particular, we prove that for each non-rational facet-defining inequality $ax \leq a_P$ for P one can construct a finite set S_a of integral vectors that satisfies $C_{S_a}(P) \subseteq (ax \leq a_P)$. As the non-rational inequality $ax \leq a_P$ cannot be facet-defining for the *rational* polyhedron $C_{S_a}(P)$, the Gomory-Chvátal cuts associated with the vectors in S_a must separate every point of the hyperplane $(ax = a_P)$ that is not contained in the maximal rational affine subspace V_R of $(ax = a_P)$. We will, in fact, show that the set of points in $(ax = a_P) \setminus V_R$ can be partitioned into a finite number of segments such that for each segment there is a single Gomory-Chvátal cut that separates all the points in the segment. The number of segments will thereby only depend on the dimension of V_R.

Our proof technique has a clear geometric interpretation. It is motivated by an observation made for rational polytopes that we illustrate as follows: Suppose that $H = (ax = a_P)$ is an integral hyperplane in $\mathbb{R}^n$. We can assume w.l.o.g. that $a_p = 0$ and that the hyperplane is defined by integral vectors $u_1, \ldots, u_{n-2}$, and v, which span a parallelepiped that does not contain any interior integral points. In other words, these vectors form a basis of the lattice defined by the integral points in H. Let $U := \text{span}(u_1, \ldots, u_{n-2})$. Then $U + \lambda v \subseteq H$, for any number λ. One can imagine that the set of integer points in H can be partitioned into subsets (or layers) associated with the parallel affine subspaces that are obtained by shifting U by some integral multiple of v (see Figure 4-6). Now consider a rational polytope P with facet $F = P \cap (ax = 0)$ such that F is a subset of $U + \{\lambda v \mid \lambda < 1\}$. Then F does not intersect the affine subspace spanned by the integer points in $U + 1 \cdot v$, but lies completely on one side of this subspace in H. Given this setting, there is some "space" between F and the next layer of integer

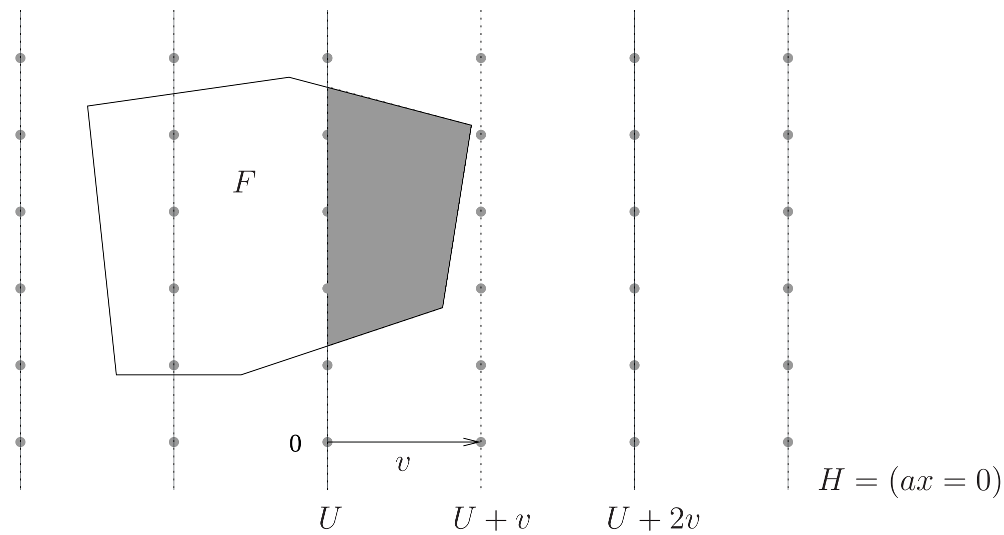

Figure 4-6: The lattice of integer points in H can be partitioned into layers that are parallel to U and obtained by shifting U by integral multiples of v. The facet $F = P \cap H$ does not intersect the affine space $U + v$. We can rotate the hyperplane H around U to obtain a Gomory-Chvátal cut for P that separates all points in the grey area.

points in H. It therefore appears intuitive, that a slight "rotation of the hyperplane H around U and in the direction of v" should result in some hyperplane $(hx = 0)$ that corresponds to a Gomory-Chvátal cut for P. Such a hyperplane would separate every point in $F \cap \big(U + \{\lambda v \mid \lambda > 0\}\big)$ and imply that $F' \subseteq U + \{\lambda v \mid \lambda \leq 0\}$. In other words, one iteration of the Gomory-Chvátal procedure would guarantee that F', and thus P', does not contain any points in H that lie strictly between the two affine subspaces U and $U+v$. Figure 4-7 illustrates the described situation in dimension two.

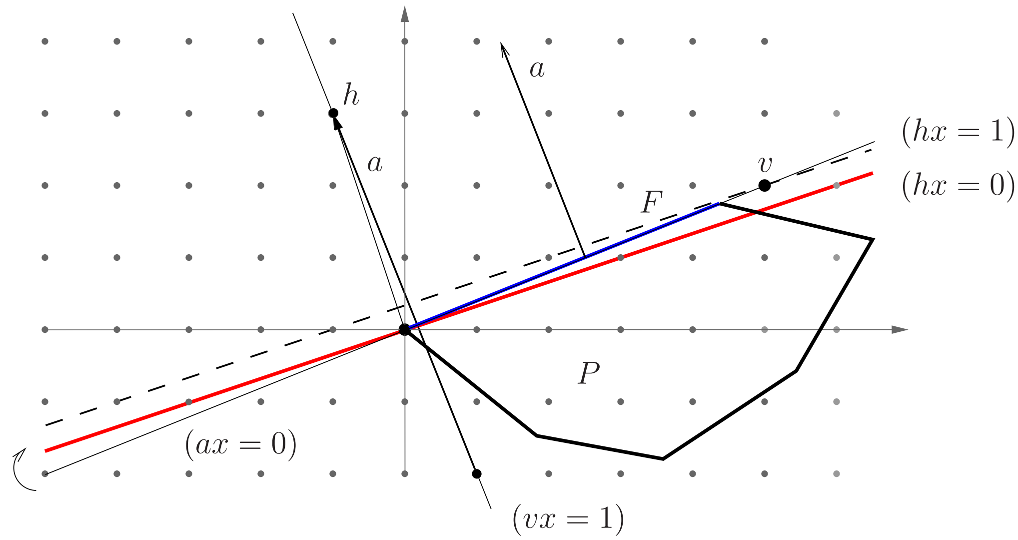

Figure 4-7: Geometric observation for rational polytopes, illustrated for $n = 2$: The hyperplane $(ax = 0)$ with $a = (-2, 5)$ is spanned by $v = (5, 2)$. Here, $U = \{0\}$. The line segment $[0, v]$ (that is, the parallelepiped spanned by v) does not contain any interior integral points, that is, $\gcd(v) = 1$. Consequently, there exists an integral vector $h_0 = (-1, 3)$ such that $h_0 v = 1$. Note that the same is true for any $h = h_0 + ka$ with $k \in \mathbb{Z}$. Hence, by choosing a large enough k, we can guarantee that there is an integral h such that $hv = 1$ and such that hx is maximized over P by a vertex in F. Since $F \subsetneq \{\lambda v \mid \lambda < 1\}$, we get that $\max\{hx \mid x \in P\} = \max\{hx \mid x \in F\} < hv = 1$, implying that $hx \leq 0$ is a Gomory-Chvátal cut for F that separates every point in $F \cap (0, v)$.

As we will formally prove in Lemma 4.6 and Corollary 4.7 below, this intuition is justified. Most importantly, it will assist in constructing Gomory-Chvátal cuts that separate the points in the non-rational parts of facets with non-rational affine hulls: In the following, we illustrate the basic idea for the special case of a non-rational facet-defining hyperplane $(ax = a_P)$ for which the maximal rational affine subspace V_R is integral and has dimension $n-2$. There is a natural generalization of this approach for the case that V_R

is non-integral or of smaller dimension. Let $F = P \cap (ax = a_P)$ be a facet of a polytope P and let us assume w.l.o.g. that $a_P = 0$. Furthermore, suppose that $(ax = 0)$ is spanned by integral vectors $u_1, \ldots, u_{n-2}$ and some non-rational vector v. Then we can approximate the hyperplane $(ax = 0)$ by a sequence of integral hyperplanes $(a^i x = 0)$ that are spanned by the vectors $u_1, \ldots, u_{n-2}$ together with an approximation $v^i \in \mathbb{Z}^n$ of the non-rational direction v. That is, the approximations also contain $U := \operatorname{span}(u_1, \ldots, u_{n-2}) = V_R$. It is intuitive that the norm of the integral vector v^i has to increase with the accuracy of the approximation, as the distance of v^i to the non-rational hyperplane $(ax = 0)$ must become smaller. Now consider the perturbation P^i of P that is obtained by replacing the non-rational facet-defining inequality $ax \leq 0$ by the approximation $a^i x \leq 0$. For large enough norm of the vector v^i, the facet $F^i = P^i \cap (a^i x = 0)$ does not intersect the affine subspace $U + v^i$ that is spanned by $n-2$ integral points in $(a^i x = 0)$. Hence, with the earlier observation regarding rational facets, there exists a Gomory-Chvátal cut $h^i x \leq 0$ for P^i that separates every point in $U + \{\lambda v^i \mid \lambda > 0\}$. Our general strategy is to utilize this cut to derive a Gomory-Chvátal cut $hx \leq 0$ for P that removes every point in $U + \{\lambda v \mid \lambda > 0\}$. Note that such h would need to have a strictly positive scalar product with v; and the maximum of hx over P would have to be attained at a vertex in F and be strictly smaller than 1. Ideally, we would want the vector h^i to satisfy these conditions. However, the modified Diophantine approximation that we use to generate the sequence of normal vectors a^i, and thus for v^i, does not guarantee these properties for every h^i. One difficulty, for example, is the fact that $h^i v^i > 0$ does not necessarily imply $hv > 0$ (see also Figure 4-8). Hence, the construction of the vector h has to balance the goal of making the scalar product hv strictly positive and making it not too large, in order to guarantee $hv < 1$.

A rather complicated construction and analysis in Lemma 4.8 will show that an integral vector h with the desired properties always exists. It gives rise to a Gomory-Chvátal cut that separates every point in the set $U + \{\lambda v \mid \lambda > 0\}$. Similarly, one can construct a cut for the non-rational part on the "other side" of U, that is, for $U - \{\lambda v \mid \lambda > 0\}$. Geometrically, the non-rational part of $(ax = 0)$ is partitioned into two pieces associated with the directions $+v$ and $-v$. The two corresponding cutting planes form a "tent" in the half-space $(ax \leq 0)$ (see Figure 4-9). In the generalization to lower-dimensional subspaces U, the non-rational part of $(ax = 0)$, spanned by non-rational vectors $v_1, \ldots, v_l$, will be partitioned into 2^l disjoint sets that correspond to parallelepipeds spanned by the

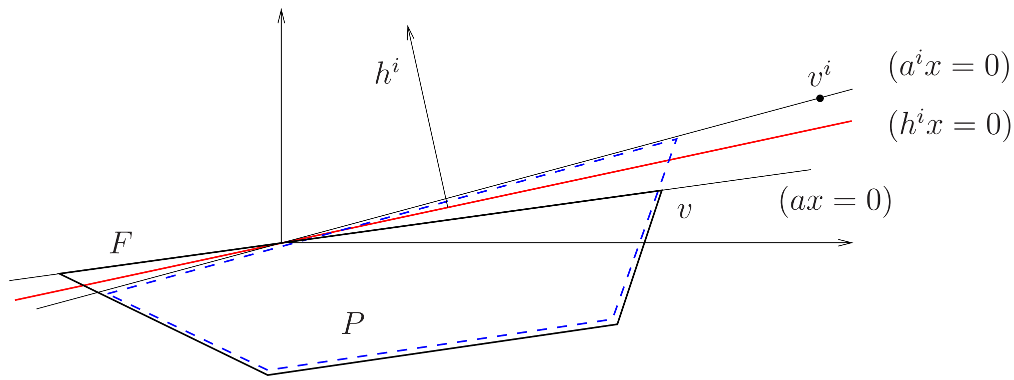

Figure 4-8: Illustration of some of the difficulties in the construction of Gomory-Chvátal cuts that separate non-rational parts of facets: $h^ix \leq 0$ is a Gomory-Chvátal cut for the approximation P^i (drawn with a dashed line) that separates every point λv^i of $F^i = P^i \cap (a^ix = 0)$ with $\lambda > 0$. However, even if the cut $h^ix \leq 0$ is also a valid Gomory-Chvátal cut for P, it does not separate any point λv with $\lambda > 0$, since $h^iv < 0$.

vectors $(\pm v_1, \ldots, \pm v_l)$.

The first lemma and corollary of this subsection formalize the observations for rational polytopes described above, which can be regarded as the geometric foundation of the proof of Step 1.

Lemma 4.6 *Let $u_1, \ldots, u_{n-2}$ and v be linearly independent vectors in $\mathbb{Z}^n$ such that*

$$\left\{ \sum_{i=1}^{n-2} \gamma_i u_i + \lambda v \,\middle|\, \gamma_i \in \mathbb{R},\ i = 1, \ldots, n-2;\ 0 < \lambda < 1 \right\} \cap \mathbb{Z}^n = \emptyset \ . \tag{4.10}$$

Then there exists a vector $y \in \mathbb{Z}^n$ such that $u_iy = 0$, for $i = 1, \ldots, n-2$, and such that $vy = 1$.

Proof. First, let us assume that the semi-open parallelepiped spanned by the vectors $u_1, \ldots, u_{n-2}$ does not contain any integral points apart from 0, that is,

$$\left\{ \sum_{i=1}^{n-2} \gamma_i u_i \,\middle|\, 0 \leq \gamma_i < 1,\ i = 1, \ldots, n-2 \right\} \cap \mathbb{Z}^n = \{0\} \ . \tag{4.11}$$

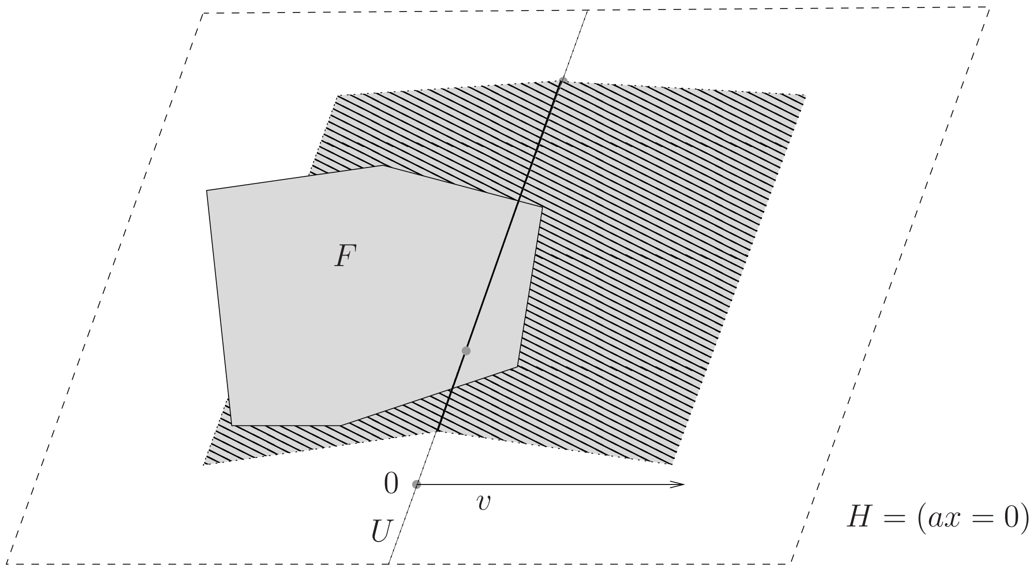

Figure 4-9: Illustration of how non-rational parts of facets are separated for $n = 3$: $H = (ax = 0)$ is a non-rational hyperplane that has a rational affine subspace U of dimension $n-2$. The non-rational direction is given by the non-rational vector v. We can construct two Gomory-Chvátal cuts that separate all points in $F \setminus U$ and that form a "tent" below H with ridge U.

Together with (4.10), we have

$$\left\{ \sum_{i=1}^{n-2} \gamma_i u_i + \lambda v \,\middle|\, 0 \le \gamma_i < 1,\ i = 1, \dots, n-2;\ 0 \le \lambda < 1 \right\} \cap \mathbb{Z}^n = \{0\} \ ,$$

that is, also the semi-open parallelepiped spanned by all $n-1$ vectors does not contain any integral points apart from 0. Now consider the system

$$Vy := \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-2} \\ v \end{bmatrix} y = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} =: b \ .$$

Note that V has full row rank and column rank $n-1$. By Theorem 2.11, there exists a unimodular matrix $U \in \mathbb{Z}^{n\times n}$, that is, $|\det(U)| = 1$, such that

$$VU = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-2} \\ v \end{bmatrix} U = \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \vdots \\ \tilde{u}_{n-2} \\ \tilde{v} \end{bmatrix} =: \left[\tilde{V} \,\middle|\, 0\right],$$

where each $\tilde{u}_i = u_i U$ and $\tilde{v} = vU$ has its n-th component zero and where $\tilde{V}$ is a non-singular integral $(n-1)\times(n-1)$ matrix. The semi-open parallelepiped spanned by the vectors $\tilde{u}_1, \ldots, \tilde{u}_{n-2}$, and $\tilde{v}$ in $(x_n = 0)$ does not contain any integral points apart from 0. Indeed, suppose there was an integral point $z = \gamma_1\tilde{u}_1 + \ldots + \gamma_{n-2}\tilde{u}_{n-2} + \lambda\tilde{v}$ with $0 \le \gamma_i < 1$, for $i = 1, \ldots, n-2$, and $0 \le \lambda < 1$, such that not all of these coefficients are zero. Then

$$zU^{-1} = \gamma_1\tilde{u}_1U^{-1} + \ldots + \gamma_{n-2}\tilde{u}_{n-2}U^{-1} + \lambda\tilde{v}U^{-1} = \gamma_1 u_1 + \ldots + \gamma_{n-2}u_{n-2} + \lambda v$$

is an integral point different from 0 in the semi-open parallelepiped spanned by $u_1, \ldots, u_{n-2}$, and v, which is a contradiction. Now observe that Lemma 2.15 implies $|\det(\tilde{V})| = 1$. Therefore, the system

$$\tilde{V}\tilde{y} = b$$

has an integral solution $\tilde{y} \in \mathbb{Z}^{n-1}$. The vector $\bar{y} = [\tilde{y}^T \; 0]^T$ satisfies $VU\bar{y} = b$ and, consequently, $y = U\bar{y}$ is an integral solution to $Vy = b$.

If assumption (4.11) is not satisfied, then we can find a set of $n-2$ integral vectors $u'_1, \ldots, u'_{n-2}$ spanning the same linear vector space as $u_1, \ldots, u_{n-2}$ such that (4.11) holds. Consequently, there is a vector $y \in \mathbb{Z}^n$ such that $u'_i y = 0$ for $i = 1, \ldots, n-2$ and $vy = 1$. Since every u_i can be written as a linear combination of the u'_i, we have $u_i y = 0$ as well. □

In the following corollary, we apply the above lemma and characterize how rational faces that satisfy a certain property behave under the Gomory-Chvátal procedure. More precisely, suppose that $H = (ax = a_P)$ is an integral hyperplane that is face-defining for a rational polytope P. Furthermore, assume that the face $F = P \cap H$ does not share any points with an $(n-2)$-dimensional affine subspace $\bar{U}$ spanned by some set of $n-1$ integral points in H. Then all points of F that lie strictly between $\bar{U}$ and the parallel affine subspace $\bar{U}'$ that is obtained by shifting $\bar{U}$ in H towards F until the next layer of integer points is touched, will be separated by a single Gomory-Chvátal cut (see Figure 4-10 for an illustration). Note that the normal vector h of such a cut has to be perpendicular to every vector in $\bar{U}$. Put differently, the hyperplane $(hx = \lfloor h_P \rfloor)$ has to be parallel to $\bar{U}$.

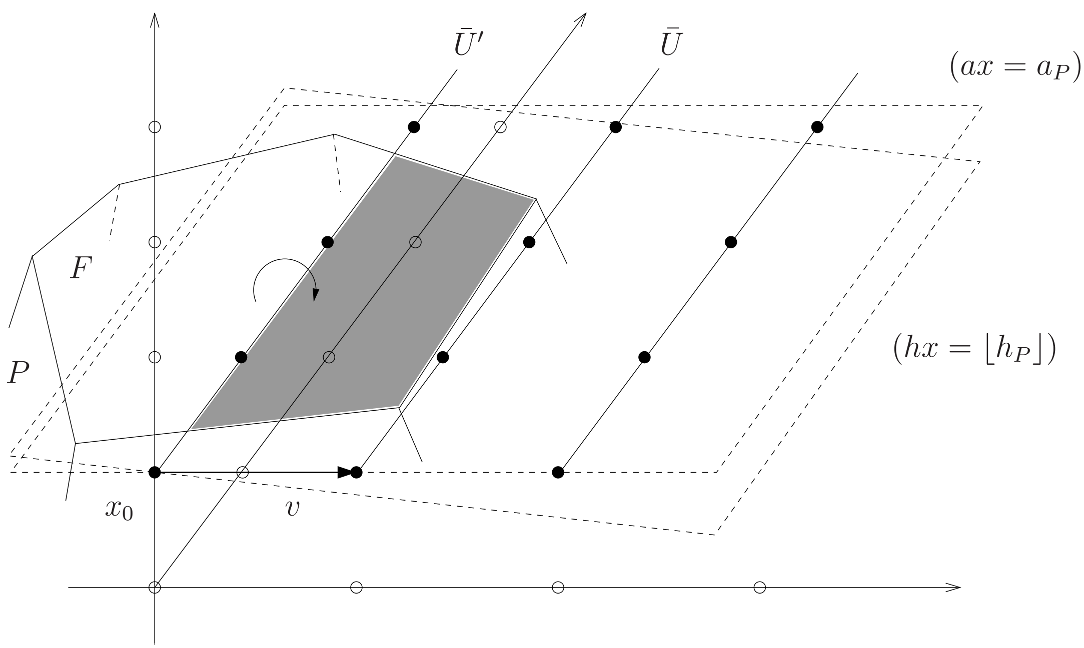

Figure 4-10: Geometric observation for rational polytopes, illustrated for $n = 3$: $H = (ax = a_P)$ is an integral hyperplane and $F = P \cap H$ is a face of the rational polytope P. Note that the integer points in H are drawn as filled black points. Since $F \cap \bar{U} = \emptyset$, there is a Gomory-Chvátal cut that separates all points in the shaded area between $\bar{U}'$ and $\bar{U}$ in H.

Corollary 4.7 *Let $H = (ax = a_P)$ be an integral hyperplane in $\mathbb{R}^n$ such that*

$$H = x_0 + \mathrm{span}(u_1, \ldots, u_{n-2}, v) \ ,$$

for vectors $u_i \in \mathbb{Z}^n$, for $i = 1, \ldots, n-2$, and vectors $x_0, v \in \mathbb{Z}^n$. Furthermore, let $P \subseteq \mathbb{R}^n$ be a polytope for which $F = P \cap (ax = a_P)$ is a face. If

(i) $\left\{x_0 + \sum_{i=1}^{n-2} \gamma_i u_i + \lambda v \,\middle|\, \gamma_i \in \mathbb{R},\ i = 1, \ldots, n-2;\ 0 < \lambda < 1\right\} \cap \mathbb{Z}^n = \emptyset$,

(ii) $F \subseteq \left\{x_0 + \sum_{i=1}^{n-2} \gamma_i u_i + \lambda v \,\middle|\, \gamma_i \in \mathbb{R},\ i = 1, \ldots, n-2;\ \lambda < 1\right\}$,

then

$$F' \subseteq \left\{ x_0 + \sum_{i=1}^{n-2} \gamma_i u_i + \lambda v \,\middle|\, \gamma_i \in \mathbb{R},\ i = 1, \ldots, n-2;\ \lambda \leq 0 \right\}.$$

Proof. We can assume w.l.o.g. that $x_0 = 0$. Because of assumption (ii), there exists an $\varepsilon \in [0, 1)$ such that

$$F \subseteq \left\{ \sum_{i=1}^{n-2} \gamma_i u_i + \lambda v \,\middle|\, \gamma_i \in \mathbb{R},\ i = 1, \ldots, n-2;\ \lambda \leq \varepsilon \right\} \ . \tag{4.12}$$

With assumption (i), Lemma 4.6 implies the existence of a vector $y \in \mathbb{Z}^n$ such that

$$u_i y = 0 \quad \text{for } i = 1, \ldots, n-2 \quad \text{and} \quad vy = 1 \ ,$$

and the same is true for any integral vector $y + ka$, where $k \in \mathbb{N}$. Now let $r_1, \ldots, r_m$ denote the set of edge directions emanating from the vertices of F to vertices of P that are not in F. Then $r_s a < 0$ for $s = 1, \ldots, m$. We can choose k large enough, so that

$$\max\left\{(y + ka)x \mid x \in P\right\} = \max\left\{(y + ka)x \mid x \in F\right\} \ .$$

Then with (4.12) and $u_i(y + ka) = 0$ for $i = 1, \ldots, n-2$, we get for arbitrary values of the coefficients γ_i

$$\max\left\{(y + ka)x \mid x \in P\right\} \leq (y + ka)\left(\sum_{i=1}^{n-2} \gamma_i u_i + \varepsilon v\right) = \varepsilon \ .$$

Consequently, $(y + ka)x \leq 0$ is a Gomory-Chvátal cut for P. Now consider any point $x \in F$ such that $x = \sum_{i=1}^{n-2} \gamma u_i + \lambda v$ with $\gamma_i \in \mathbb{R}$ and $\lambda > 0$. Then

$$(y + ka)\, x = \sum_{i=1}^{n-2} \gamma_i (y + ka)\, u_i + \lambda(y + ka)v = \lambda > 0 \ .$$

Hence, the point x violates the Gomory-Chvátal cut $(y+ka)x \leq 0$ and, therefore, $x \notin F'$.

□

While the above lemma and corollary concern integral hyperplanes, in the remainder of this subsection we will focus on affine spaces that cannot be described by rational data. Lemma 4.8 below can be seen as the core of the proof of Step 1. Therein, we establish for every non-rational vector space $V = (ax = 0)$ the existence of sequences of vectors and numbers, which satisfy a distinct list of properties. The sequences are associated with integral approximations of the non-rational hyperplane V. The starting point in the construction of these sequences is a special type of Diophantine approximation $\{a^i\}$ of the non-rational normal vector a. If $u_1, \ldots, u_k$ denote a maximal set of integral and linearly independent vectors in V, then the normal vectors $a^i \in \mathbb{Z}^n$ are perpendicular to each of the vectors $u_1, \ldots, u_k$. As a result, the approximations $(a^i x = 0)$ of the hyperplane V contain the maximal rational subspace $V_R = \text{span}(u_1, \ldots, u_k)$ of V. In particular, $(ax = 0) \cap (a^i x = 0) = V_R$. Each integral hyperplane $(a^i x = 0)$ is spanned by the vectors $u_1, \ldots, u_k$ together with $l = n - 1 - k$ additional integral vectors, denoted by $v_1^i, \ldots, v_l^i$, which can be regarded as approximations of the non-rational directions of V. These vectors will be chosen very carefully among the infinite number of possible sets of vectors spanning $(a^i x = 0)$, as not all choices will guarantee the properties that we require for the other sequences and numbers derived from them. Most importantly, they will be *almost* orthogonal to one another. The vectors v_j^i give rise to non-rational vectors w_j^i that span the non-rational part of $(ax = 0)$. More precisely, each w_j^i is obtained as projection of the vector v_j^i onto $(ax = 0)$, scaled by a factor, so that all w_j^i have a same given length. We refer to Figure 4-11 for an illustration. As the quality of the approximations of V increases with the index i, the w_j^i's will, at some point, also be almost orthogonal to one another. This property of the w_j^i's will turn out to be material in the subsequent proof of Step 1. Apart from the mentioned sequences a^i, v_j^i, and w_j^i, which have very natural geometric interpretations, we also establish a sequence of integral vectors $h^i(\delta)$, for each $\delta \in \{-1,1\}^l$, whose construction is more involved. They arise as integral linear combinations of the integral vectors found in Lemma 4.6, which were the basis for Gomory-Chvátal cuts separating points in rational facets between affine layers of integral points (see Figure 4-10 and Corollary 4.7). Some of the properties that these vectors satisfy are as follows: Each $h^i(\delta)$ is perpendicular to the vectors $u_1, \ldots, u_k$ and, therefore, the hyperplane $(h^i x = 0)$ is parallel to V_R. Moreover, the scalar product of $h^i(\delta)$ with each non-rational vector $\delta_j w^i_j$ is strictly positive, but very small.

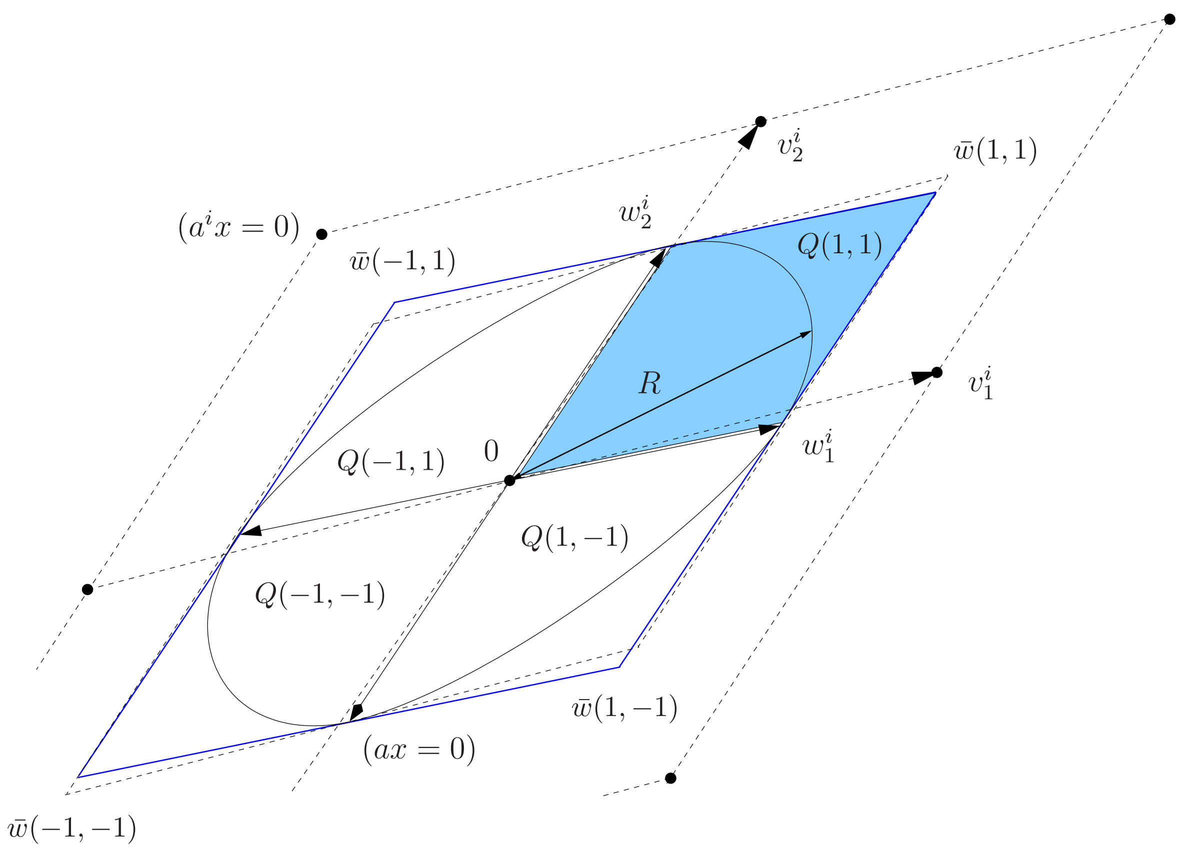

Figure 4-11: Illustration of Lemma 4.8 for $n = 3$ and $\dim(V_R) = 0$: The non-rational hyperplane $(ax = 0)$ is approximated by integral hyperplanes $(a^i x = 0)$. The integral vectors v^i_1 and v^i_2 span $(a^i x = 0)$ and are almost orthogonal to each other. Their directions give rise to non-rational vectors w^i_1 and w^i_2 of a given length R in $(ax = 0)$. For each parallelepiped $Q(\delta)$ spanned by vectors $\delta_1 w^i_1$ and $\delta_2 w^i_2$, with $\delta \in \{-1,1\}^2$, there exists a Gomory-Chvátal cut that separates every point in $Q(\delta) \setminus V_R$.

To understand the motivation behind these properties, let us consider the non-rational parallelepiped $Q(\delta)$ that is spanned by $u_1, \ldots, u_k$ and the non-rational vectors $\delta_1 w_1^i, \ldots, \delta_l w_l^i$. When maximizing $h^i(\delta)$ over $Q(\delta)$, the maximum is attained at $\bar{w}(\delta) = \delta_1 w_1^i + \ldots + \delta_l w_l^i$, or any other point in $Q(\delta)$ that can be written as $\bar{w}(\delta) + u$ for some $u \in V_R$. Moreover, the properties of $h^i(\delta)$ guarantee that $0 < h^i(\delta)\bar{w}(\delta) < 1$. As a consequence, $h^i(\delta)x \leq 0$ is a Gomory-Chvátal cut for $Q(\delta)$, and this cut implies that $\big(Q(\delta)\big)' \subseteq V_R$. Thus, for the special case that the non-rational polytope is the $(n-1)$-dimensional parallelepiped $Q(\delta)$ or, contained in it, the single integral vector $h^i(\delta)$ implies a finite set S with the properties that we are looking for in Step 1 of the proof.

For a general polytope P with a facet $F = P \cap (ax = 0)$, we can cover F by at most 2^l parallelepipeds in V. Then every vector $h^i(\delta)$ will give rise to a Gomory-Chvátal cut that separates all the points in corresponding parallelepiped that do not belong to V_R. Note that for this, we also need the property that, when $h^i(\delta)$ is maximized over P, the maximum is attained at a vertex in F. In other words, every vector $h^i(\delta)$ must have a non-positive scalar product with the directions of edges connecting a vertex in F and a vertex outside of F. Indeed, we construct the $h^i(\delta)$ in Lemma 4.8 with the requirement that for a given finite set of vectors $r_1, \ldots, r_m$, their scalar product with these vectors is nonpositive.

The proof of Lemma 4.8 strongly relies on properties of reduced bases of integral lattices.

Lemma 4.8 *Let $R > 0$ be a constant and let $V \subseteq \mathbb{R}^n$ be a non-rational hyperplane through the origin, that is, $V = (ax = 0)$ for some $a \in \mathbb{R}^n \setminus \mathbb{Q}^n$. Let U be the maximal rational subspace of V and assume that U is spanned by vectors $u_1, \ldots, u_k \in \mathbb{Z}^n$, that is, $\dim(U) = k$, where $0 \leq k \leq n-2$. Let $l := n-k-1$. Furthermore, let $r_1, \ldots, r_m \in \mathbb{R}^n$ such that for $s = 1, \ldots, m$,*

$$r_s\, a < 0 \ . \tag{4.13}$$

Then there exists a constant $c > 0$ only depending on l such that there exist sequences

$$\{a^i\} \subseteq \mathbb{Z}^n, \quad \{v_1^i\}, \ldots, \{v_l^i\} \subseteq \mathbb{Z}^n, \quad \{q_1^i\}, \ldots, \{q_l^i\} \subseteq \mathbb{R}, \quad \{w_1^i\}, \ldots, \{w_l^i\} \subseteq \mathbb{R}^n$$

that satisfy the following properties:

(i) $\gcd(a^i) = 1$.

(ii) $r_s\, a^i \leq 0$ *for* $s = 1, \ldots, m$.

(iii) $\|a^i\|\, \|\bar{a}^i - \bar{a}\| \longrightarrow 0$, *where* $\bar{a}^i = a^i / \|a^i\|$ *and* $\bar{a} = a / \|a\|$.

(iv) $(a^i x = 0) = \operatorname{span}(u_1, \ldots, u_k, v_1^i, \ldots, v_l^i)$.

(v) $\left\|v_j^i\right\| \longrightarrow \infty$.

(vi) $\left\|\frac{v_j^i}{q_j^i} - w_j^i\right\| \longrightarrow 0$.

(vii) $\left\|w_j^i\right\| = R$.

(viii) $V = \operatorname{span}(u_1, \ldots, u_k, w_1^i, \ldots, w_l^i)$.

(ix) $\left\|\tilde{w}_j^i\right\| \geq c\, R$, *where* $\tilde{w}_j^i$ *denotes the orthogonal projection of* w_j^i *onto* $\left(\operatorname{span}(u_1, \ldots, u_k, w_1^i, \ldots, w_{j-1}^i)\right)^{\perp}$.

(x) There exists a constant $C > 0$ *such that for any* $\varepsilon > 0$, *there exists an integer* $N(\varepsilon)$ *such that for all* $i \geq N(\varepsilon)$ *and for all* $\alpha = (\alpha_1, \ldots, \alpha_l) \in \mathbb{R}_+^l$ *with* $\|\alpha\|_\infty \leq 1$, *there exist vectors* $\{h_\alpha^i(\delta)\} \subseteq \mathbb{Z}^n$ *for all* $\delta \in \{-1, 1\}^l$ *such that*

$$\begin{array}{rcll} h_\alpha^i(\delta) & \perp & u_p & \text{for} \quad p = 1, \ldots, k \\ \alpha_j - \varepsilon & \leq & \delta_j w_j^i\, h_\alpha^i(\delta) \leq \alpha_j + \varepsilon & \text{for} \quad j = 1, \ldots, l \\ \delta_j h_\alpha^i(\delta)\, v_j^i & = & \left\lfloor \alpha_j q_j^i \right\rfloor & \text{for} \quad j = 1, \ldots, l \\ 0 & \geq & r_s\, h_\alpha^i(\delta) & \text{for} \quad s = 1, \ldots, m \\ |h_\alpha^i(\delta)\, a^i| & \leq & C\, \|a^i\|^2 & . \end{array}$$

Proof. Let us assume w.l.o.g. that the vectors $u_1, \ldots, u_k$ form a basis of the lattice $U \cap \mathbb{Z}^n$. If this is not the case, we can replace the original vectors by another set of vectors in U that has this property. Let V_{IR} denote the set of points in V that are not contained in the maximal rational subspace of V, that is, $V_{IR} := V \setminus U$. Let $\{a^i\} \subseteq \mathbb{Z}^n$ be a sequence of vectors according to Corollary 4.4 such that for $p = 1, \dots, k$,

$$a^i \perp u_p$$

and

$$\left\|a^i\right\| \left\|\bar{a}^i - \bar{a}\right\| \longrightarrow 0 \ . \tag{4.14}$$

We can assume w.l.o.g. that $\gcd(a^i) = 1$, since the same properties hold if we divide a^i by some positive integer. Thus, the sequence $\{a^i\}$ satisfies properties (i) and (iii). Furthermore,(4.14) implies for $s = 1, \dots, m$,

$$\left|r_s \bar{a}^i - r_s \bar{a}\right| \longrightarrow 0 \ .$$

As $r_s \bar{a} < 0$ by assumption (4.13), there exists some constant $\beta > 0$ such that $r_s \bar{a}^i \leq -\beta$ for large enough i. Hence, noting that $\|a^i\| \longrightarrow \infty$ because of $a \in \mathbb{R}^n \setminus \mathbb{Q}^n$, it also holds that for $s = 1, \dots, m$ and large enough i,

$$r_s a^i \leq -\beta \ . \tag{4.15}$$

In particular, property (ii) is guaranteed for large enough i.

Let $\Lambda^i = (a^i x = 0) \cap \mathbb{Z}^n$ denote the lattice defined by the integer points in the integral hyperplane $(a^i x = 0)$. In the following claim, we show that norm of the shortest vector in $\Lambda^i \setminus U$ grows with i.

Claim 4.9 *Let z^i denote a shortest vector in $\Lambda^i \setminus U$. Then $\|z^i\| \longrightarrow \infty$.*

Proof of claim. Suppose that there exists some positive constant K such that for all i, one can find a point $z^i \in \Lambda^i \setminus U$ with $\|z^i\| \leq K$. Let $\mathrm{proj}(z^i)$ denote the projection of z^i onto the hyperplane $(ax = 0)$, that is, $\mathrm{proj}(z^i) + \lambda a = z^i$, where $\lambda = (az^i)/\left\|a\right\|^2$. As $z^i \notin (ax = 0)$, we have $\|z^i - \mathrm{proj}(z^i)\| > 0$. Furthermore, since the number of integer points in $B(0, K)$ is finite, there must exist some positive number D such that $\|z^i - \text{proj}(z^i)\| \geq D$ for every i. However, using $\bar{a}^i z^i = 0$ and (4.14), we get

$$\left\|z^i - \text{proj}(z^i)\right\| = |\lambda| \, \|a\| = \frac{|az^i|}{\|a\|} = \left|\bar{a}z^i\right| = \left|\bar{a}z^i - \bar{a}^i z^i\right| \leq \left\|\bar{a} - \bar{a}^i\right\| \; K \longrightarrow 0 \;\; ,$$

which is a contradiction. $\square$

Claim 4.9 implies that, for sufficiently large i, we can assume for every $v \in \Lambda^i \setminus U$,

$$\|v\|^2 \geq \frac{1}{2}\left(\sum_{p=1}^{k} \|u_p\|\right)^2 \;\; .$$

Since $(a^i x = 0)$ is an integral hyperplane and $U \subseteq (a^i x = 0)$, we can find integral vectors $v_1^i, \ldots, v_l^i$ according to Theorem 4.5. That is,

$$(a^i x = 0) = \text{span}(u_1, \ldots, u_k, v_1^i, \ldots, v_l^i) \;\; ,$$

and the vectors $u_1, \ldots, u_k, v_1^i, \ldots, v_l^i$ form a basis of the lattice Λ^i. Furthermore, let $\tilde{v}_1^i$ be the orthogonal projection of v_1^i onto $U^\perp$ and let $\tilde{v}_j^i$ denote the orthogonal projection of v_j^i onto $\big(\text{span}(u_1, \ldots, u_k, v_1^i, \ldots, v_{j-1}^i)\big)^\perp$, for $j = 2, \ldots, l$. Then it also holds by Theorem 4.5 that for $j = 1, \ldots, l$,

$$\left\|\tilde{v}_j^i\right\| \geq c_1 \left\|v_j^i\right\| \;\; , \tag{4.16}$$

where c_1 is a constant that only depends on l. Note that property (iv) of the lemma follows. Furthermore, observe that $v_j^i \in \Lambda^i \setminus U$ for $j = 1, \ldots, l$. Consequently, Claim 4.9 implies property (v).

Since $u_1, \ldots, u_k, v_1^i, \ldots, v_l^i$ form a basis of Λ^i, we have for every $s \in \{1, \ldots, l\}$,

$$\left\{ \sum_{p=1}^{k} \gamma_p u_p + \sum_{j=1}^{l} \lambda_j v_j^i \,\middle|\, \gamma_p \in \mathbb{R},\ p = 1, \ldots, k;\ \lambda_j \in \mathbb{R},\ j = 1, \ldots, l;\ 0 < \lambda_s < 1 \right\} \cap \mathbb{Z}^n = \emptyset \;\; . \tag{4.17}$$

Indeed, if this was not the case and there was a point $z \in \mathbb{Z}^n$ such that

$$z = \sum_{p=1}^{k} \gamma_p u_p + \sum_{j=1}^{l} \lambda_j v_j^i$$

and such that $0 < \lambda_s < 1$, then

$$z' = \sum_{p=1}^{k} \left(\gamma_p - \lfloor \gamma_p \rfloor\right) u_p + \sum_{j=1}^{l} \left(\lambda_j - \lfloor \lambda_j \rfloor\right) v_j^i \in \left(\mathbb{Z}^n \cap \Pi(u_1, \ldots, u_k, v_1^i, \ldots, v_l^i)\right) \ .$$

Hence, z' is an integral point in the semi-open parallelepiped spanned by the basis vectors. Because of $0 < \lambda_s < 1$, it holds that $z' \neq 0$, and this cannot be true.

Now, let us define for $j = 1, \ldots, l$,

$$(w_j^i, q_j^i) := \arg\min \left\{ \left\| \frac{1}{q} v_j^i - w \right\| \ \middle| \ w \in V_{IR};\ \|w\| = R;\ q \in \mathbb{R}_+ \right\} \ . \tag{4.18}$$

Intuitively, w_j^i is the closest point in the intersection of V_{IR} with the ball $B(0, R)$ to the line spanned by v_j^i. The definition of w_j^i immediately implies property (vii) of the lemma. In the following claim, we show property (vi).

Claim 4.10 *For $j = 1, \ldots, l$, we have $q_j^i \longrightarrow \infty$ and*

$$\left\| \frac{v_j^i}{q_j^i} - w_j^i \right\| \longrightarrow 0 \ .$$

Proof of claim. We first show the second part. Let w denote the projection of the point $(R\,\bar{v}_j^i)$ onto the non-rational hyperplane $(ax = 0)$, where $\bar{v}_j^i = v_j^i / \left\|v_j^i\right\|$. We have $w = R\bar{v}_j^i - \lambda a$, where $\lambda = (aR\bar{v}_j^i)/\left\|a\right\|^2$. Furthermore, let $q = \left\|v_j^i\right\|/R > 0$. Note that for $\bar{w} = w/\left\|w\right\|$, it holds that $R\bar{w} \in V_{IR}$ and $\|R\bar{w}\| = R$. Therefore, $(R\bar{w}, q)$ is a feasible pair in the minimization (4.18) that defines (w_j^i, q_j^i). Consequently,

$$\begin{aligned}
\left\| \frac{v_j^i}{q_j^i} - w_j^i \right\| &\leq \left\| \frac{v_j^i}{q} - R\,\bar{w} \right\| = \left\| \frac{v_j^i}{\left\|v_j^i\right\|/R} - R\,\bar{w} \right\| = \left\| R\bar{v}_j^i - R\bar{w} + (w - w) \right\| \\
&\leq \left\| R\bar{v}_j^i - w \right\| + \left\| w - R\bar{w} \right\| \ .
\end{aligned}$$

We get, using $\bar{a}^i \bar{v}_j^i = 0$ and (4.14), that

$$\left\| R\bar{v}_j^i - w \right\| = |\lambda| \left\|a\right\| = \left| R\bar{a}\bar{v}_j^i \right| = R \left| \bar{a}\bar{v}_j^i - \bar{a}^i \bar{v}_j^i \right| \leq R \left\| \bar{a} - \bar{a}^i \right\| \left\| \bar{v}_j^i \right\| = R \left\| \bar{a} - \bar{a}^i \right\| \longrightarrow 0 \ .$$

This also implies that $\|w\| \longrightarrow R$ and, therefore, the second part of the claim holds. The first part, $q_j^i \longrightarrow \infty$, follows from $\left\|v_j^i\right\| \longrightarrow \infty$, $\left\|w_j^i\right\| = R$, and $\left\|v_j^i/q_j^i - w_j^i\right\| \longrightarrow 0$. $\square$

Next, we prove property (ix). By (4.16), we have for $j = 1, \ldots, l$,

$$\frac{1}{q_j^i}\left\|\tilde{v}_j^i\right\| \geq \frac{1}{q_j^i} c_1 \left\|v_j^i\right\| \quad . \tag{4.19}$$

Let $\tilde{w}_j^i$ denote the orthogonal projection of w_j^i onto $\left(\operatorname{span}(u_1, \ldots, u_k, w_1^i, \ldots, w_{j-1}^i)\right)^\perp$, for $j = 1, \ldots, l$. Because of Claim 4.10, there is for every $\tau > 0$ a number $N(\tau)$, such that for all $i \geq N(\tau)$,

$$\frac{\left\|v_j^i\right\|}{q_j^i} - \tau \leq \left\|w_j^i\right\| \leq \frac{\left\|v_j^i\right\|}{q_j^i} + \tau$$

and

$$\frac{\left\|\tilde{v}_j^i\right\|}{q_j^i} - \tau \leq \left\|\tilde{w}_j^i\right\| \leq \frac{\left\|\tilde{v}_j^i\right\|}{q_j^i} + \tau \quad .$$

Now let γ be some small constant such that $c_1 > \gamma > 0$. By (4.19),

$$\frac{\left\|\tilde{v}_j^i\right\|}{q_j^i} - (c_1 - \gamma)\frac{\left\|v_j^i\right\|}{q_j^i} \geq \gamma \frac{\left\|v_j^i\right\|}{q_j^i} \quad .$$

Using this observation and $R = \left\|w_j^i\right\|$, we obtain

$$\begin{aligned} \left\|\tilde{w}_j^i\right\| - (c_1 - \gamma)R &\geq \frac{\left\|\tilde{v}_j^i\right\|}{q_j^i} - \tau - (c_1 - \gamma)\left(\frac{\left\|v_j^i\right\|}{q_j^i} + \tau\right) \\ &\geq \gamma\frac{\left\|v_j^i\right\|}{q_j^i} - \tau - (c_1 - \gamma)\tau \\ &\geq \gamma(R - \tau) - \tau - (c_1 - \gamma)\tau \quad . \end{aligned}$$

Note that we can choose τ small enough such that the last expression is nonnegative. Hence, $c = (c_1 - \delta) > 0$ is the desired constant for property (ix). Since c_1 only depends on l, the same is true for c.

Now observe that property (ix) implies that the vectors $u_1, \ldots, u_k, w_1^i, \ldots, w_l^i$ are linearly independent. This is because $\left\|\tilde{w}_j^i\right\| > 0$ for $j = 1, \ldots, l$ and

$$\operatorname{span}(u_1, \ldots, u_k, w_1^i, \ldots, w_l^i) = \operatorname{span}(\tilde{u}_1, \ldots, \tilde{u}_k, \tilde{w}_1^i, \ldots, \tilde{w}_l^i) \ ,$$

where $\tilde{u}_1 = u_1$, and where for $p = 2, \ldots, k$, the vector $\tilde{u}_p$ denotes the orthogonal projection of u_p onto $(\operatorname{span}(u_1, \ldots, u_{p-1}))^\perp$. Consequently, property (viii) is satisfied.

In the remainder of the proof, we show property (x). Because of (4.17) and Lemma 4.6, there exists for each $s = 1, \ldots, l$ a vector $y_s^i \in \mathbb{Z}^n$ such that

$$y_s^i \in (u_1 x = 0) \cap \ldots \cap (u_k x = 0) \cap \bigcap_{j \neq s} (v_j^i x = 0) \cap (v_s^i x = 1) =: L_s^i \ . \tag{4.20}$$

Since L_s^i is the intersection of $n-1$ linearly independent hyperplanes in $\mathbb{R}^n$, it is a line. Because $a^i \perp u_j$ and $a^i \perp v_j^i$, the direction of the line is a^i. Let us assume w.l.o.g. that $a_1 \neq 0$, and therefore $a_1^i \neq 0$ for large enough i. Let $\bar{y}_s^i$ denote the intersection of L_s^i with the hyperplane $(x_1 = 0)$. Note that $\bar{y}_s^i \neq \pm\infty$ because of the assumption $a_1^i \neq 0$. That is, $\bar{y}_s^i$ is the unique solution to the system

$$\begin{bmatrix} e_1 \\ u_1 \\ \vdots \\ u_k \\ v_1^i \\ \vdots \\ v_{s-1}^i \\ v_s^i \\ v_{s+1}^i \\ \vdots \\ v_l^i \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} .$$

For convenience, we introduce some additional notation: Let U denote the matrix with rows u_p, $p = 1, \ldots, k$, and let V_{-s}^i denote the matrix with rows v_j^i for all $j = 1, \ldots, l$ such that $j \neq s$. Similarly, let W_{-s}^i denote the matrix with rows w_j^i for all $j = 1, \ldots, l$ with $j \neq s$. Finally, let V^i and W^i denote the matrices with rows v_j^i and w_j^i, for $j = 1, \ldots, l$, respectively. Then the above system becomes

$$\begin{bmatrix} e_1 \\ U \\ V^i_{-s} \\ v^i_s \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Claim 4.11 *For $s = 1, \ldots, l$, $\bar{y}^i_s q^i_s \longrightarrow \bar{x}^i_s$, where $\bar{x}^i_s$ denotes the unique solution to the linear system of equations*

$$\begin{bmatrix} e_1 \\ U \\ W^i_{-s} \\ w^i_s \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Proof of claim. Let v^i_j/q^i_j denote the vector obtained by dividing every component of v^i_j with the scalar q^i_j. Furthermore, let V^i_{-s}/q^i_{-s} be the matrix with rows v^i_j/q^i_j for all $j \neq s$. Then

$$\begin{bmatrix} e_1 \\ U \\ V^i_{-s} \\ v^i_s \end{bmatrix} \bar{y}^i_s = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} e_1 \\ U \\ V^i_{-s} \\ v^i_s/q^i_s \end{bmatrix} \bar{y}^i_s q^i_s = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} e_1 \\ U \\ V^i_{-s}/q^i_{-s} \\ v^i_s/q^i_s \end{bmatrix} \bar{y}^i_s q^i_s = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

where Claim 4.10 implies

$$\begin{bmatrix} e_1 \\ U \\ V^i_{-s}/q^i_{-s} \\ v^i_s/q^i_s \end{bmatrix} \longrightarrow \begin{bmatrix} e_1 \\ U \\ W^i_{-s} \\ w^i_s \end{bmatrix}.$$

□

Now we will show that the entries of $\bar{x}^i_s$ cannot become arbitrarily large.

Claim 4.12 *There exists a constant $K_1 > 0$ such that for sufficiently large i,*

$$\|\bar{x}_s^i\|_\infty \leq K_1 \ .$$

Proof of claim. By definition,

$$\bar{x}_s^i = \begin{bmatrix} e_1 \\ U \\ W_{-s}^i \\ w_s^i \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \ .$$

Therefore, it suffices to show that the entries of the inverse matrix in the above equation cannot be arbitrarily large. We have

$$\left(A^i\right)^{-1} := \begin{bmatrix} e_1 \\ U \\ W^i \end{bmatrix}^{-1} = \frac{1}{\det(A^i)} \operatorname{adj}(A^i) \ ,$$

where $\operatorname{adj}(A^i)$ denotes the adjugate matrix of A^i. Since all entries of A^i are bounded (note that $\left\|w_j^i\right\| = R$), every entry of $\operatorname{adj}(A^i)$ is bounded as well. Hence, it is sufficient to show that $|\det(A^i)|$ can be bounded from below for large enough i. The absolute value of the determinant of A^i corresponds to the volume of the parallelepiped spanned by the vectors $u_1, \ldots, u_k, w_1^i, \ldots, w_l^i, e_1$ (see also Section 2.2.3). Therefore, it holds that

$$\left|\det(A^i)\right| = \|\tilde{u}_1\| \ldots \|\tilde{u}_k\| \left\|\tilde{w}_1^i\right\| \ldots \left\|\tilde{w}_l^i\right\| \|\tilde{e}_1\| \ .$$

Here, $\tilde{e}_1$ denotes the orthogonal projection of e_1 onto $(\operatorname{span}(u_1, \ldots, u_k, w_1^i, \ldots, w_l^i))^\perp$. Hence, by property (viii), the vector $\tilde{e}_1$ is the orthogonal projection of e_1 onto $V^\perp$. Because of the assumption $a_1 \neq 0$, it follows that $\|\tilde{e}_1\| > 0$. With property (ix), we obtain for sufficiently large i,

$$\left|\det(A^i)\right| \geq (cR)^l \ \|\tilde{u}_1\| \ldots \|\tilde{u}_k\| \|\tilde{e}_1\| \ .$$

As the expression on the right in the last inequality is a strictly positive constant, the claim follows. $\square$

Now let us define for any vector $M^i = (M_1^i, \ldots, M_l^i) \in \mathbb{N}^l$, the set

$$L^i(M^i) := (u_1 x = 0) \cap \ldots \cap (u_k x = 0) \cap (v_1^i x = M_1^i) \cap \ldots \cap (v_l^i x = M_l^i) \ .$$

By virtue of (4.20), we have

$$\left(M_1^i y_1^i + \ldots + M_l^i y_l^i\right) \in L^i(M^i) \cap \mathbb{Z}^n$$

and therefore,

$$L^i(M^i) \cap \mathbb{Z}^n \neq \emptyset \ .$$

Moreover, $L^i(M^i)$ is a line with direction a^i that intersects the hyperplane $(x_1 = 0)$ in $\bar{y}^i(M^i) := M_1^i \, \bar{y}_1^i + \ldots + M_l^i \, \bar{y}_l^i$. Therefore, we can write

$$L^i(M^i) = \left\{ x \in \mathbb{R}^n \,\middle|\, x = \bar{y}^i(M^i) + \mu a^i, \ \mu \in \mathbb{R} \right\} \ .$$

Observe that every line segment of length $\|a^i\|$ of $L^i(M^i)$ must contain an integral point.

Now let $\varepsilon > 0$ be an arbitrary constant. In the remainder of the proof we show that there exists a constant C and an integer $N(\varepsilon)$ such that for all $i \geq N(\varepsilon)$ and for every $\alpha \in \mathbb{R}_+^l$ with $\|\alpha\|_\infty \leq 1$, there is a vector $M^i = M^i(\alpha) \in \mathbb{N}^l$ and a number $\mu_0^i = \mu_0^i(\alpha) \in \mathbb{R}$ such that for each $\mu \in [\mu_0^i, \mu_0^i + 1]$ and each $\delta \in \{-1, 1\}^l$ the vector

$$h^i(\delta M^i, \mu) := \bar{y}^i(\delta M^i) + \mu a^i = \delta_1 M_1^i \, \bar{y}_1^i + \ldots + \delta_l M_l^i \, \bar{y}_l^i + \mu \, a^i \tag{4.21}$$

satisfies

$$\begin{aligned}
h^i(\delta M^i, \mu) \ &\perp \ u_p && \text{for } p = 1, \ldots, k && (4.22)\\
\alpha_j - \varepsilon \ &\leq \ \delta_j w_j^i \, h^i(\delta M^i, \mu) \leq \alpha_j + \varepsilon && \text{for } j = 1, \ldots, l && (4.23)\\
\delta_j h^i(\delta M^i, \mu) \, v_j^i \ &= \ \left\lfloor \alpha_j q_j^i \right\rfloor && \text{for } j = 1, \ldots, l && (4.24)\\
0 \ &\geq \ r_s \, h^i(\delta M^i, \mu) && \text{for } s = 1, \ldots, m && (4.25)\\
\left| h^i(\delta M^i, \mu) \, a^i \right| \ &\leq \ C \left\| a^i \right\|^2 \ . && && (4.26)
\end{aligned}$$

Here, the notation δM^i means $(\delta_1 M_1^i, \ldots, \delta_l M_l^i)$. Since every line segment of length $\|a^i\|$ contains an integral point, there must exists some $\mu \in [\mu_0^i, \mu_0^i + 1]$ such that $h^i(\delta M^i, \mu)$

is an integral vector. Consequently, this would imply property (x) of the lemma.

First, observe that condition (4.22) always holds, since any h of the form (4.21) is a linear combination of vectors that are perpendicular to the vectors u_p, $p = 1, \ldots, k$. Using definition (4.21) of $h^i(\delta M^i, \mu)$, condition (4.25) becomes

$$0 \geq \delta_1 M_1^i \, r_s \, \bar{y}_1^i + \ldots + \delta_l M_l^i \, r_s \, \bar{y}_l^i + \mu \, r_s a^i \ .$$

Now let $\beta > 0$ be the constant from (4.15), that is, $r_s \, a^i \leq -\beta$, for $s = 1, \ldots, m$. Then (4.25) becomes

$$\mu \geq \frac{\delta_1 M_1^i \, r_s \, \bar{y}_1^i + \ldots + \delta_l M_l^i \, r_s \, \bar{y}_l^i}{-r_s a^i} \ ,$$

and for

$$\mu_0^i := \max_{s=1,\ldots,m} \left\{ \frac{\delta_1 M_1^i \, r_s \, \bar{y}_1^i + \ldots + \delta_l M_l^i \, r_s \, \bar{y}_l^i}{\beta} \right\} \ ,$$

this condition is satisfied for all $\mu \geq \mu_0^i$. Let $r^i \in \{r_1, \ldots, r_m\}$ such that

$$\mu_0^i = \frac{\delta_1 M_1^i \, r^i \, \bar{y}_1^i + \ldots + \delta_l M_l^i \, r^i \, \bar{y}_l^i}{\beta} \ .$$

Then by (4.21),

$$h^i(\delta M^i, \mu_0^i) = \delta_1 M_1^i \, \left(\bar{y}_1^i + \frac{1}{\beta} r^i \, \bar{y}_1^i \, a^i \right) + \ldots + \delta_l M_l^i \, \left(\bar{y}_l^i + \frac{1}{\beta} r^i \, \bar{y}_l^i \, a^i \right) \ ,$$

and (4.23) becomes for $\mu = \mu_0^i$ and $j = s$,

$$\alpha_s - \varepsilon \leq \delta_s \left(\sum_{j=1}^{l} \delta_j M_j^i \, \left(w_s^i \, \bar{y}_j^i + \frac{1}{\beta} \, r^i \, \bar{y}_j^i \, w_s^i \, a^i \right) \right) \leq \alpha_s + \varepsilon \ . \tag{4.27}$$

Now let us define $M_j^i := \lfloor \alpha_j q_j^i \rfloor$. Note that $M^i \in \mathbb{N}^l$. In the following, we will show that this choice for M^i satisfies (4.27). For this, we consider the terms in (4.27) separately. We start with the terms $\delta_s \delta_j M_j^i \, w_s^i \, \bar{y}_j^i$. If $j = s$, then we get with Claims 4.10 ($q_s^i \to \infty$)

and 4.11 ($w_s^i \bar{x}_s^i = 1$),

$$\begin{aligned}
\left|\delta_s \delta_s M_s^i \, w_s^i \, \bar{y}_s^i - \alpha_s\right| &= \left|\left\lfloor \alpha_s q_s^i \right\rfloor w_s^i \, \bar{y}_s^i - \alpha_s\right| \leq \left|\alpha_s w_s^i \, \bar{y}_s^i q_s^i - \alpha_s\right| + \left|w_s^i \, \bar{y}_s^i\right| \\
&= \left|\alpha_s w_s^i \, \bar{y}_s^i q_s^i - \alpha_s w_s^i \bar{x}_s^i\right| + \frac{1}{q_s^i}\left|w_s^i \, \bar{y}_s^i q_s^i - w_s^i \, \bar{x}_s^i + w_s^i \, \bar{x}_s^i\right| \\
&\leq \left|w_s^i \left(\bar{y}_s^i q_s^i - \bar{x}_s^i\right)\right| + \frac{1}{q_s^i}\left|w_s^i \left(\bar{y}_s^i q_s^i - \bar{x}_s^i\right)\right| + \frac{1}{q_s^i}\left|w_s^i \, \bar{x}_s^i\right| \\
&\leq \left(1 + \frac{1}{q_s^i}\right)\left\|w_s^i\right\|\left\|\bar{y}_s^i q_s^i - \bar{x}_s^i\right\| + \frac{1}{q_s^i} \\
&= R\left(1 + \frac{1}{q_s^i}\right)\left\|\bar{y}_s^i q_s^i - \bar{x}_s^i\right\| + \frac{1}{q_s^i} \longrightarrow 0 \ .
\end{aligned}$$

Hence, for every $\varepsilon_1 > 0$ and for sufficiently large i,

$$\alpha_s - \varepsilon_1 \leq \delta_s \delta_s M_s^i \, w_s^i \, \bar{y}_s^i \leq \alpha_s + \varepsilon_1 \ . \tag{4.28}$$

For $j \neq s$, it similarly follows by Claims 4.10 ($q_j^i \to \infty$) and 4.11 ($w_s^i \bar{x}_j^i = 0$) that

$$\begin{aligned}
\left|\delta_s \delta_j M_j^i \, w_s^i \, \bar{y}_j^i\right| &= \left|\left\lfloor \alpha_j q_j^i \right\rfloor w_s^i \, \bar{y}_j^i\right| = \left|\alpha_j w_s^i \, \bar{y}_j^i q_j^i - \alpha_j w_s^i \bar{x}_j^i\right| + \left|w_s^i \, \bar{y}_j^i\right| \\
&\leq \left|w_s^i \left(\bar{y}_j^i q_j^i - \bar{x}_j^i\right)\right| + \frac{1}{q_j^i}\left|w_s^i \left(\bar{y}_j^i q_j^i - \bar{x}_j^i\right)\right| \\
&\leq R\left(1 + \frac{1}{q_j^i}\right)\left\|\bar{y}_j^i q_j^i - \bar{x}_j^i\right\| \longrightarrow 0 \ .
\end{aligned}$$

Thus, for sufficiently large i,

$$-\varepsilon_1 \leq \delta_s \delta_j M_j^i \, w_s^i \, \bar{y}_j^i \leq \varepsilon_1 \ . \tag{4.29}$$

Now consider the terms $\delta_s\delta_j M_j^i \frac{1}{\beta} r^i\,\bar{y}_j^i\,w_s^i\,a^i$. With Claim 4.11, we obtain

$$
\begin{aligned}
\left|\delta_s\delta_j M_j^i \frac{1}{\beta} r^i\,\bar{y}_j^i\,w_s^i\,a^i\right| &= \frac{1}{\beta}\left|\lfloor\alpha_j q_j^i\rfloor\,(r^i\,\bar{y}_j^i)\,(w_s^i\,a^i)\right| \\
&\leq \frac{1}{\beta}\left|\alpha_j\,(r^i\,\bar{y}_j^i\,q_j^i)\,(w_s^i\,a^i)\right| + \frac{1}{\beta}\left|(r^i\,\bar{y}_j^i)\,(w_s^i\,a^i)\right| \\
&\leq \frac{1}{\beta}\left|r^i\,\bar{y}_j^i\,q_j^i\right|\left|w_s^i\,a^i\right| + \frac{1}{\beta}\left|r^i\,\bar{y}_j^i\right|\left|w_s^i\,a^i\right| \\
&= \frac{1}{\beta}\left(1+\frac{1}{q_j^i}\right)\left|r^i\,\bar{y}_j^i\,q_j^i\right|\left|w_s^i\,a^i\right| \\
&= \frac{1}{\beta}\left(1+\frac{1}{q_j^i}\right)\left|r^i\,\bar{y}_j^i\,q_j^i - r^i\bar{x}_j^i + r^i\bar{x}_j^i\right|\left|w_s^i\,a^i\right| \\
&\leq \frac{1}{\beta}\left(1+\frac{1}{q_j^i}\right)\Big(\left|r^i\,(\bar{y}_j^i\,q_j^i - \bar{x}_j^i)\right| + \left|r^i\,\bar{x}_j^i\right|\Big)\left|w_s^i\,a^i\right| \\
&\leq \frac{1}{\beta}\left(1+\frac{1}{q_j^i}\right)\left\|r^i\right\|\Big(\left\|\bar{y}_j^i q_j^i - \bar{x}_j^i\right\| + \left\|\bar{x}_j^i\right\|\Big)\left|w_s^i\,a^i\right| \quad .
\end{aligned}
$$

We can bound

$$
\frac{1}{\beta}\left(1+\frac{1}{q_j^i}\right)\left\|r^i\right\|\Big(\left\|\bar{y}_j^i q_j^i - \bar{x}_j^i\right\| + \left\|\bar{x}_j^i\right\|\Big)
$$

from above by Claims 4.10, 4.11, and 4.12 for sufficiently large i. Furthermore, using (4.14) and $\bar{a}w_s^i = 0$, we get for each $s = 1, \ldots, l$,

$$
\left|w_s^i a^i\right| = \left\|a^i\right\|\left|\bar{a}^i w_s^i\right| = \left\|a^i\right\|\left|\bar{a}^i w_s^i - \bar{a}w_s^i\right| \leq \left\|a^i\right\|\left\|\bar{a}^i - \bar{a}\right\|\left\|w_s^i\right\| = R\left\|a^i\right\|\left\|\bar{a}^i - \bar{a}\right\| \longrightarrow 0 \quad . \tag{4.30}
$$

It follows that

$$
\left|\delta_s\delta_j M_j^i \frac{1}{\beta} r^i\,\bar{y}_j^i\,w_s^i\,a^i\right| \longrightarrow 0 \quad .
$$

Consequently, for large enough i,

$$
-\varepsilon_1 \leq \delta_s\delta_j M_j^i \frac{1}{\beta} r^i\,\bar{y}_j^i\,w_s^i\,a^i \leq \varepsilon_1 \quad . \tag{4.31}
$$

Plugging (4.28),(4.29), and (4.31) into (4.27), we can bound $\delta_s w_s^i\, h^i(\delta M^i, \mu_0^i)$ for large enough i by

$$\alpha_s - 2\,l\,\varepsilon_1 \leq \delta_s w_s^i\, h^i(\delta M^i, \mu_0^i) \leq \alpha_s + 2\,l\,\varepsilon_1 \ .$$

Because of (4.30), we have $|w_s^i a^i| \leq \varepsilon_1$ for large enough i. Therefore, for all $\mu \in [\mu_0^i, \mu_0^i{+}1]$,

$$\alpha_s - (2l+1)\,\varepsilon_1 \leq \delta_s w_s^i\, h^i(\delta M^i, \mu) \leq \alpha_s + (2l+1)\,\varepsilon_1 \ .$$

Finally, by choosing $\varepsilon_1 < \frac{\varepsilon}{(2l+1)}$, we obtain that there exists some integer $N(\varepsilon)$ such that for all $i \geq N(\varepsilon)$ and for all $\mu \in [\mu_0^i, \mu_0^i + 1]$,

$$\alpha_s - \varepsilon \leq \delta_s w_s^i\, h^i(\delta M^i, \mu) \leq \alpha_s + \varepsilon \ .$$

In particular, there must exist an integral $h_\alpha^i(\delta)$ with this property. Note that $N(\varepsilon)$ does not depend on α. This proves condition (4.23). For condition (4.24), observe that $hv_j^i = M_j^i$ for every $h \in L^i(M^i)$ and every $j = 1, \ldots, l$. We thus get

$$\delta_j h^i(\delta M^i, \mu) v_j^i = \delta_j^2 M_j^i = \lfloor \alpha_j q_j^i \rfloor \ .$$

Finally, consider condition (4.26). For every $\mu \in [\mu_0^i, \mu_0^i + 1]$, we have

$$\begin{aligned}
\left|h(\delta M^i, \mu) a^i\right| &\leq \left|h(\delta M^i, \mu_0^i)\, a^i\right| + \left|a^i a^i\right| \\
&= \left|\sum_{j=1}^{l} \delta_j \left\lfloor a_j q_j^i \right\rfloor \left(\bar{y}_j^i + \frac{1}{\beta}(r^i \bar{y}_j^i) a^i\right) a^i\right| + \left\|a^i\right\|^2 \\
&\leq \sum_{j=1}^{l} \left(\left|\alpha_j q_j^i \left(\bar{y}_j^i\, a^i + \frac{1}{\beta}\, r^i\, \bar{y}_j^i\, a^i\, a^i\right)\right| + \left|\bar{y}_j^i\, a^i + \frac{1}{\beta}\, r^i\, \bar{y}_j^i\, a^i\, a^i\right| \right) + \left\|a^i\right\|^2 \\
&\leq \sum_{j=1}^{l} \left(1 + \frac{1}{q_j^i}\right) \left(\left|\bar{y}_j^i q_j^i\, a^i\right| + \frac{1}{\beta} \left|r^i\, \bar{y}_j^i q_j^i\, a^i\, a^i\right| \right) + \left\|a^i\right\|^2 \\
&\leq \sum_{j=1}^{l} \left(1 + \frac{1}{q_j^i}\right) \left\|\bar{y}_j^i q_j^i\right\| \left(\left\|a^i\right\| + \frac{1}{\beta} \left\|r^i\right\| \left\|a^i\right\|^2 \right) + \left\|a^i\right\|^2 \ .
\end{aligned}$$

Since $\left\|\bar{y}_j^i q_j^i\right\|$ is bounded because of Claims 4.11 and 4.12, since $q_j^i \to \infty$ by Claim 4.10, and since $\|r^i\|$ is bounded as well, condition (4.26) is satisfied for sufficiently large i. $\square$

In the proof above, we chose the vectors $v_1^i, \ldots, v_l^i$, which span together with the vectors $u_1, \ldots, u_k$ the integral approximation $(a^i x = 0)$ of $(ax = 0)$, in a specific way. The vectors $u_1, \ldots, u_k, v_1^i, \ldots, v_l^i$ form a basis of the lattice of integer points in $(a^i x = 0)$ and they satisfy properties that are characteristic for reduced bases. In other words, the vectors $v_1^i, \ldots, v_l^i$ are almost perpendicular to each other. We already leveraged this special property when we showed property (ix) of Lemma 4.8 and also in the proof of Claim 4.12, which was required for the final analysis in the lemma. However, there is a second reason why an arbitrary choice of these vectors would not allow us to prove the main result of this section. Recall that our goal is to show that for each non-rational facet-defining inequality $ax \leq a_P$ of a polytope P, there exists a finite set $S_a \subseteq \mathbb{Z}^n$ such that $C_{S_a}(P) \subseteq (ax \leq a_P)$. To illustrate why reduced bases are crucial, consider the special case that $a_P = 0$ and $(ax = 0) \cap \mathbb{Q}^n = \{0\}$. The basic geometric motivation behind the construction in Lemma 4.8 arose from the objective to cover $F = P \cap (ax = 0)$ with at most 2^{n-1} parallelepipeds, spanned by the vectors $\delta_1 w_1^i, \ldots, \delta_{n-1} w_{n-1}^i$, where $\delta \in \{-1, 1\}^{n-1}$. Indeed, if these parallelepipeds covered F, then the vectors $h^i(\delta)$ of Lemma 4.8 gave rise to Gomory-Chvátal cuts that separate every point in F apart from 0: This is because we can choose the vectors r_s for Lemma 4.8 such that

$$\max\left\{h^i(\delta)x \mid x \in P\right\} = \max\left\{h^i(\delta)x \mid x \in F\right\} \quad .$$

Then, for an appropriate choice of the parameters α and ε in (x), we can achieve

$$\max\left\{h^i(\delta)x \mid x \in P\right\} \leq h^i(\delta)\left(\delta_1 w_1^i + \ldots + \delta_{n-1} w_{n-1}^i\right) < 1$$

and, consequently, $h^i(\delta)x \leq 0$ is a Gomory-Chvátal cut for P. As this is true for every $\delta \in \{-1, 1\}^{n-1}$, these 2^{n-1} Gomory-Chvátal cuts imply $F' = \{0\}$.

Since P is bounded, F is contained in some ball of radius R around the origin. Clearly, if the $w_1^i, \ldots, w_{n-1}^i$ are orthogonal to each other and of length R, then the parallelepipeds cover this ball and therefore F (see Figure 4-12).

However, the smaller the angles between the vectors $w_1^i, \ldots, w_{n-1}^i$, the longer the w_j^i's

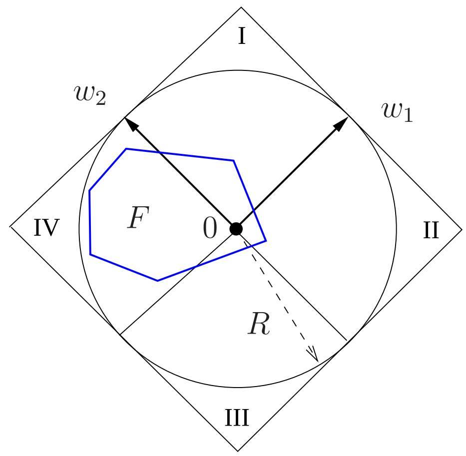

Figure 4-12: Illustration of why reduced bases play a crucial role: The facet $F = P \cap (ax = 0)$ is contained in $B(0, R)$. If w_1^i and w_2^i are orthogonal to each other and of length R, the four parallelepipeds spanned by the vectors $\pm w_1^i$ and $\pm w_2^i$ cover F.

have to be to guarantee that F is completely covered (see Figure 4-13). The analysis in Lemma 4.8 required that the w_j^i's have a fixed length, which is chosen at the beginning of the construction. However, as the lattices of integer points in $\text{span}(v_1^i, \ldots, v_{n-1}^i)$ change with every index i, arbitrary bases of the lattices would result in arbitrary angles between the w_j^i's. Therefore, it is not certain that any fixed length R would guarantee the covering property that is envisioned. By choosing reduced bases, we make sure that the v_j^i's and, hence, the w_j^i's are almost orthogonal to each other. Moreover, their orthogonality defect only depends on the dimension. As a consequence, we can choose a certain fixed length for the vectors w_j^i (that is, the value R in Lemma 4.8) that only depends on the radius of the ball that fits F and the dimension n.

In the next lemma, we utilize the sequences from Lemma 4.8 to prove that for every non-rational facet-defining inequality $ax \leq a_P$ of a polytope P, there exists a finite set S_a of integral vectors such that $C_{S_a}(P) \subseteq (ax \leq a_P)$. This property immediately implies the existence of a finite set $S \subset \mathbb{Z}^n$ with $C_S(P) \subseteq P$.

Lemma 4.13 *Let $(a, a_P) \in \mathbb{R}^{n+1} \backslash \mathbb{Q}^{n+1}$ such that $(ax = a_P)$ is a non-rational hyperplane with $P \subseteq (ax \leq a_P)$ and $P \cap (ax = a_P) \neq \emptyset$. Then there exists a finite set $S \subseteq \mathbb{Z}^n$ such that $C_S(P) \subseteq (ax \leq a_P)$.*

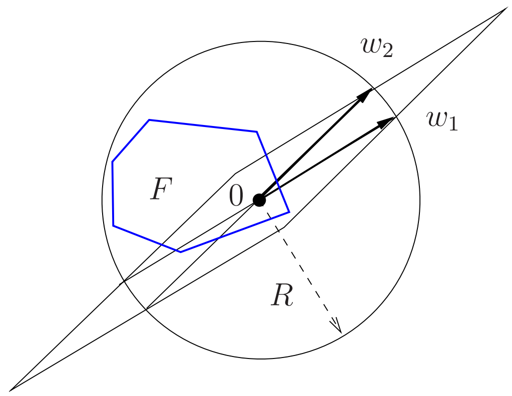

Figure 4-13: Illustration of why reduced bases play a crucial role: The facet $F = P \cap (ax = 0)$ is contained in $B(0, R)$. If w_1^i and w_2^i are of length R, but the angle between them is very small, the four parallelepipeds spanned by the vectors $\pm w_1^i$ and $\pm w_2^i$ do not cover F.

Proof. There are three possible types of non-rational inequalities $ax \leq a_P$:

(a) $a \in \mathbb{Q}^n$ and $a_P \in \mathbb{R} \setminus \mathbb{Q}$.

(b) $a \in \mathbb{R}^n \setminus \mathbb{Q}^n$ and $(ax = a_P) \cap \mathbb{Q}^n \neq \emptyset$.

(c) $a \in \mathbb{R}^n \setminus \mathbb{Q}^n$ and $(ax = a_P) \cap \mathbb{Q}^n = \emptyset$.

In the following, we consider each case separately:

Case (a): If $a \in \mathbb{Q}^n$, then we can assume w.l.o.g. that $a \in \mathbb{Z}^n$ by scaling (a, a_P) by some rational number, if necessary. Consequently, $ax \leq \lfloor a_P \rfloor$ is a Gomory-Chvátal cut for P and $(ax \leq \lfloor a_P \rfloor) \subseteq (ax \leq a_P)$. Then $S = \{a\}$ has the desired property and we are done.

In the following, let us assume that $a \in \mathbb{R}^n \setminus \mathbb{Q}^n$ and that the same is true for every λa with $\lambda \in \mathbb{R}$. Let $F = P \cap (ax = a_P)$ and let $r_1, \ldots, r_m \in \mathbb{R}^n$ denote the set of edge directions emanating from the vertices of F to vertices of P that are not in F. Note that $r_s\, a < 0$, for $s = 1, \ldots, m$. First, consider case (b):

Case (b): Let V_R denote the maximal rational affine subspace contained in $(ax = a_P)$ and let $u_1, \ldots, u_k \in \mathbb{Z}^n$ and $x_0 \in (ax = a_P) \cap \mathbb{Q}^n$ such that $V_R = x_0 + \operatorname{span}(u_1, \ldots, u_k)$. Define $l := n - k - 1$ and $U := \operatorname{span}(u_1, \ldots, u_k)$. Note that $U = \{0\}$ is possible. Since P is bounded, there exists an $R_1 > 0$ such that for every $x \in F$ there is an $u \in U$ with

$$x \in x_0 + u + B(0, R_1) \ . \tag{4.32}$$

Let $p_0 \in \mathbb{Z}^n$ and let $q_0 \geq 1$ be an integer such that $x_0 = p_0/q_0$. Furthermore, let c be the constant from property (ix) in Lemma 4.8 and let c_1 be the constant from Lemma 4.1. Let us fix a constant R such that $R \geq R_1 c_1 / c$ and consider the sequences that exist according to Lemma 4.8 for $V = (ax = 0)$ and R.

First, observe that we can choose i large enough such that $a^i x \leq \lfloor a^i x_0 \rfloor$ is a Gomory-Chvátal cut for P: Property (ii) in Lemma 4.8 implies

$$\max \left\{ a^i x \mid x \in P \right\} = \max \left\{ a^i x \mid x \in F \right\} = a^i x_0 + \max \left\{ a^i (x - x_0) \mid x \in F \right\} \ ,$$

and by property (iii) and the boundedness of P, we have for all $x \in F$,

$$\begin{aligned} \left\| a^i (x - x_0) \right\| &= \left\| a^i \right\| \left\| \bar{a}^i (x - x_0) \right\| = \left\| a^i \right\| \left\| (\bar{a}^i - \bar{a})(x - x_0) \right\| \\ &\leq \left\| a^i \right\| \left\| \bar{a}^i - \bar{a} \right\| \left\| x - x_0 \right\| \longrightarrow 0 \ . \end{aligned}$$

Hence, we can choose i large enough such that

$$\max \left\{ a^i x \mid x \in P \right\} \leq a^i x_0 + \frac{1}{2q_0} \ ,$$

which implies that $a^i x \leq \lfloor a^i x_0 \rfloor$ is a Gomory-Chvátal cut for P.

Now let $\alpha = \frac{1}{2q_0(l+1)}(1, \ldots, 1)$. Also by Lemma 4.8, there exists an index i such that the vectors $v_j := v_j^i$ and $w_j := w_j^i$, for $j = 1, \ldots, l$, and the integral vectors $h(\delta) := h_\alpha^i(\delta)$, for each $\delta \in \{-1,1\}^l$, satisfy

$$h(\delta) \perp u_p \qquad \text{for } p = 1,\dots,k \tag{4.33}$$

$$0 < \delta_j w_j \, h(\delta) \le (q_0(l+1))^{-1} \qquad \text{for } j = 1,\dots,l \tag{4.34}$$

$$\delta_j h(\delta)\, v_j \ge 1 \qquad \text{for } j = 1,\dots,l \tag{4.35}$$

$$0 \ge r_s\, h(\delta) \qquad \text{for } s = 1,\dots,m \ . \tag{4.36}$$

Moreover, it holds that $\|\tilde{w}_j\| \ge cR$, where $\tilde{w}_j$ denotes the orthogonal projection of w_j onto $(\mathrm{span}(u_1,\dots,u_k,w_1,\dots,w_{j-1}))^\perp$. Observe that by (4.32), every point $x \in F$ can be written as

$$x = x_0 + u' + \sum_{j=1}^{l} \tilde{\lambda}_j \tilde{w}_j \ ,$$

where $u' \in U$ and $|\tilde{\lambda}_j| \le R_1/(cR)$, for $j = 1,\dots,l$. Then it follows by Lemma 4.1 that every $x \in F$ can be expressed as

$$x = x_0 + u + \sum_{j=1}^{l} \lambda_j w_j \ , \tag{4.37}$$

where $u \in U$ and $|\lambda_j| \le c_1 R_1/(cR) \le 1$, for $j = 1,\dots,l$. Now consider any $\delta \in \{-1,1\}^l$. We obtain with (4.33)-(4.36) and (4.37),

$$\begin{aligned}
\max\{h(\delta)\,x \mid x \in P\} &= \max\{h(\delta)\,x \mid x \in F\} \le h(\delta)\,x_0 + \sum_{j=1}^{l} \max_{\lambda_j \in [-1,1]} \{\lambda_j h(\delta)\, w_j\} \\
&= h(\delta)\,x_0 + \sum_{j=1}^{l} \delta_j h(\delta)\, w_j \le h(\delta)\,x_0 + l(q_0(l+1))^{-1} \\
&< h(\delta)\,x_0 + \frac{1}{q_0} \le \lfloor h(\delta)\,x_0 \rfloor + 1 \ .
\end{aligned}$$

Hence, $h(\delta)\,x \le \lfloor h(\delta)\,x_0 \rfloor$ is a Gomory-Chvátal cut for P for every $\delta \in \{-1,1\}^l$. Now consider an arbitrary $x \in (ax = a_P) \setminus V_R$. By (4.37), there exists an $u \in U$ and $\lambda_j \in \mathbb{R}_+$ and $\delta_j \in \{-1,1\}$ for $j = 1,\dots,l$ such that

$$x = x_0 + u + \sum_{j=1}^{l} \lambda_j \delta_j w_j \ .$$

Note that $\sum_{j=1}^{l} \lambda_j > 0$, as $x \notin V_R$. Consequently,

$$h(\delta)\, x = h(\delta)\, x_0 + \sum_{j=1}^{l} \lambda_j \delta_j h(\delta)\, w_j > h(\delta)\, x_0 \geq \lfloor h(\delta)\, x_0 \rfloor \quad ,$$

that is, x violates the Gomory-Chvátal cut $h(\delta)\, x \leq \lfloor h(\delta)\, x_0 \rfloor$. Now let H denote the polyhedron defined by the intersection of the 2^l half-spaces associated with the Gomory-Chvátal cuts $h(\delta)\, x \leq \lfloor h(\delta)\, x_0 \rfloor$, with $\delta \in \{-1, 1\}^l$. Then by the last observation,

$$\Big((ax = a_P) \cap H\Big) = \Big((ax = a_P) \cap \bigcap_{\delta \in \{-1,1\}^l} \big(h(\delta)\, x \leq \lfloor h(\delta)\, x_0 \rfloor\,\big)\Big) \subseteq V_R \quad . \tag{4.38}$$

Similarly, let us consider the integral hyperplane $(a^i x = a^i x_0)$. Any $x \in (a^i x = a^i x_0) \setminus V_R$ can be written as

$$x = x_0 + u + \sum_{j=1}^{l} \lambda_j \delta_j v_j^i \quad ,$$

for some $u \in U$, and $\lambda_j \in \mathbb{R}_+$ and $\delta_j \in \{-1, 1\}$, $j = 1, \ldots, l$; and in this representation it must also hold that $\sum_{j=1}^{l} \lambda_j > 0$. Then with (4.35)

$$h(\delta)\, x = h(\delta)\, x_0 + \sum_{j=1}^{l} \lambda_j \delta_j h(\delta)\, v_j > h(\delta)\, x_0 \geq \lfloor h(\delta)\, x_0 \rfloor \quad .$$

This implies that also every point in $(a^i x = a^i x_0) \setminus V_R$ is separated by some Gomory-Chvátal cut $h(\delta)\, x \leq \lfloor h(\delta)\, x_0 \rfloor$ and, thus,

$$\Big((a^i x = a^i x_0) \cap H\Big) \subseteq V_R \quad . \tag{4.39}$$

As every hyperplane $\big(h(\delta)\, x = \lfloor h(\delta)\, x_0 \rfloor\,\big)$ is parallel to V_R, either every point in V_R satisfies the corresponding inequality or every point in V_R violates it. Therefore,

$$\Big((ax = a_P) \cap H\Big) = \Big((a^i x = a^i x_0) \cap H\Big) \in \{\emptyset, V_R\} \quad .$$

Observe furthermore that every minimal face of $\big((a^i x \leq a^i x_0) \cap H\big)$ is also a minimal face of $\big((ax \le a_P) \cap H\big)$ and vice versa (see also Theorem 2.4). Consequently,

$$\Big(\big(a^i x \le \lfloor a^i x_0 \rfloor\big) \cap H\Big) \subseteq \Big(\big(a^i x \le a^i x_0\big) \cap H\Big) = \Big(\big(ax \le a_P\big) \cap H\Big) \subseteq (ax \le a_P) \ .$$

It follows that a^i and the vectors $h(\delta)$, for $\delta \in \{-1,1\}^l$, form the desired set S of the lemma.

Case (c): In the remainder of the proof, we consider the case $(ax = a_P) \cap \mathbb{Q}^n = \emptyset$. Let $u_1, \ldots, u_k \in \mathbb{Z}^n$ be a maximal set of linearly independent integral vectors such that $au_i = 0$ for $i = 1, \ldots, k$. Let $U := \operatorname{span}(u_1, \ldots, u_k)$ and note that $U = \{0\}$ is possible. Furthermore, take an arbitrary point $x_0 \in F$. Since P is bounded, there exists a constant $R_1 > 0$ such that for every $x \in F$ there is an $u \in U$ such that

$$x \in x_0 + u + B(0, R_1) \ . \tag{4.40}$$

Let us fix an $R \ge R_1 c_1 / c$, where c and c_1 are the constants from property (ix) in Lemma 4.8 and Lemma 4.1, respectively. Now consider the sequences that exist according to Lemma 4.8 for $V = (ax = 0)$ and R .

Property (ii) from Lemma 4.8 implies that if there exists an index i and an integer a_0^i such that

$$a_0^i + 1 > \max\left\{a^i x \mid x \in P\right\} = \max\left\{a^i x \mid x \in F\right\} \ge \min\left\{a^i x \mid x \in F\right\} > a_0^i \ ,$$

then $a^i x \le a_0^i$ is a Gomory-Chvátal cut for P with the property that every point in F violates the cut and such that

$$(a^i x \le a_0^i) \cap F = \emptyset \ .$$

In particular, one can then find an $\varepsilon_1 > 0$ such that $\big(P \cap (a^i x \le a_0^i)\big) \subseteq (ax \le a_P - \varepsilon_1)$. This implies that there exists a rational polyhedron $Q \supseteq P$ such that $(a^i x \le a_0^i)$ is also a Gomory-Chvátal cut for Q and such that $Q \cap (a^i x \le a_0^i) \subseteq (ax \le a_P)$. The facet normals of Q together with a^i imply the desired set S of the lemma.

Let us assume in the remainder of the proof of part (c) that for every i, there exists an integer a_0^i such that

$$F \cap (a^i x = a_0^i) \ne \emptyset \ .$$

Let $y^i \in F \cap (a^i x = a_0^i)$. Since $\gcd(a^i) = 1$ according to property (i) of Lemma 4.8, there exists an $z_0^i \in (a^i x = a_0^i) \cap \mathbb{Z}^n$. We have

$$(a^i x = a_0^i) = z_0^i + \operatorname{span}(u_1, \ldots, u_k, v_1^i, \ldots, v_l^i) \ .$$

Let $\tilde{x}_0^i$ denote the projection of x_0 onto the hyperplane $(a^i x = a_0^i)$, that is,

$$\tilde{x}_0^i = x_0 + \frac{a_0^i - a^i x_0}{\|a^i\|^2}\, a^i \ . \tag{4.41}$$

Note that because of property (iii) in Lemma 4.8 and the boundedness of P,

$$\begin{aligned} \left|a_0^i - a^i x_0\right| &= \left|a^i y^i - a^i x_0\right| = \left\|a^i\right\| \left|\bar{a}^i y^i - \bar{a}^i x_0 + (\bar{a} x_0 - \bar{a} y^i)\right| \\ &\le \left\|a^i\right\| \left\|\bar{a}^i - \bar{a}\right\| \left\|x_0 - y^i\right\| \longrightarrow 0 \ . \end{aligned} \tag{4.42}$$

We can assume w.l.o.g. that the point $z_0^i \in (a^i x = a_0^i) \cap \mathbb{Z}^n$ is chosen such that there exist numbers $\gamma_1^i, \ldots, \gamma_k^i, \mu_1^i, \ldots, \mu_l^i \in [0,1]$ such that

$$\tilde{x}_0^i = z_0^i + \gamma_1^i u_1 + \ldots + \gamma_k^i u_k + \mu_1^i v_1^i + \ldots + \mu_l^i v_l^i \ . \tag{4.43}$$

This is because $\tilde{x}_0^i \in (a^i x = a_0^i)$ can be written as

$$\begin{aligned} \tilde{x}_0^i &= z_0^i + \sum_{p=1}^{k} \gamma_p^i u_p + \sum_{j=1}^{l} \mu_j^i v_j^i \\ &= \left(z_0^i + \sum_{p=1}^{k} \lfloor \gamma_p^i \rfloor u_p + \sum_{j=1}^{l} \lfloor \mu_j^i \rfloor v_j^i\right) + \sum_{p=1}^{k} (\gamma_p^i - \lfloor \gamma_p^i \rfloor) u_p + \sum_{j=1}^{l} (\mu_j^i - \lfloor \mu_j^i \rfloor) v_j^i \ , \end{aligned}$$

and

$$\left(z_0^i + \sum_{p=1}^{k} \lfloor \gamma_p^i \rfloor u_p + \sum_{j=1}^{l} \lfloor \mu_j^i \rfloor v_j^i\right) \in \left(a^i x = a_0^i\right) \cap \mathbb{Z}^n \ .$$

Figure 4-14 illustrated the described situation.

Next, we show that $\tilde{x}_0^i$, and therefore also x_0, is far away from any integer point in the hyperplane $\left(a^i x = a_0^i\right)$.

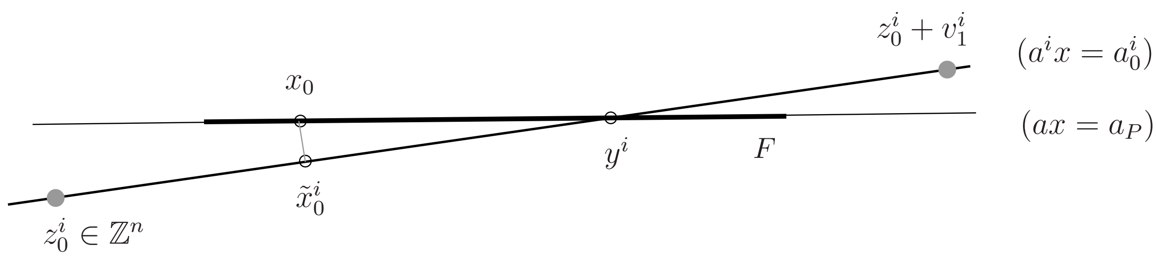

Figure 4-14: Illustration of the situation in part (c) of the proof of Lemma 4.13 in the special case that for every i there exists an integer a_0^i such that $F \cap (a^i x = a_0^i) \neq \emptyset$.

Claim 4.14 *Any vertex f^i of the parallelepiped*

$$z_0^i + \bar{\Pi}(u_1, \ldots, u_k, v_1^i, \ldots, v_l^i)$$

satisfies $\|x_0 - f^i\| \longrightarrow \infty$.

Proof of claim. As F is bounded and as x_0 and y^i are points in F, there exists a constant K_1 such that for all i, $\|x_0 - y^i\| \leq K_1$. Then

$$\left\|f^i - y^i\right\| = \left\|f^i - x_0 + x_0 - y^i\right\| \leq \left\|f^i - x_0\right\| + \left\|x_0 - y^i\right\| \leq \left\|f^i - x_0\right\| + K_1$$

implies $\|f^i - x_0\| \geq \|f^i - y^i\| - K_1$. Hence, in order to show the claim it suffices to prove $\|f^i - y^i\| \longrightarrow \infty$. Suppose that there exists some positive constant $K_2 > 0$ such that for all i we have $\|f^i - y^i\| \leq K_2$. Note that then $f^i \in B(x_0, K_1 + K_2) \cap \mathbb{Z}^n$. Let $\tilde{f}^i$ denote the projection of f^i onto the hyperplane $(ax = a_P)$, that is, $\tilde{f}^i + \lambda a = f^i$, where $\lambda = (af^i - a_P)/\|a\|^2 = (af^i - ay^i)/\|a\|^2$. Since $f^i \in \mathbb{Z}^n$ and $f^i \notin (ax = a_P)$ (remember that $(ax = a_P) \cap \mathbb{Q}^n = \emptyset$) and since the number of integer points in $B(x_0, K_1 + K_2)$ is finite, there must exist some positive number D such that $\left\|f^i - \tilde{f}^i\right\| \geq D$, for every i. However, with property (iii) from Lemma 4.8 and using $\bar{a}^i(f^i - y^i) = 0$, we get

$$\begin{aligned} \left\|f^i - \tilde{f}^i\right\| &= |\lambda| \, \|a\| = \frac{|af^i - ay^i|}{\|a\|} = \left|\bar{a}(f^i - y^i) - \bar{a}^i(f^i - y^i)\right| \\ &\leq \left\|\bar{a} - \bar{a}^i\right\| \left\|f^i - y^i\right\| \leq K_2 \left\|\bar{a} - \bar{a}^i\right\| \longrightarrow 0 \ , \end{aligned}$$

which is a contradiction. $\square$

As the above claim implies that $\tilde{x}_0^i$ is far away from any integer point in the hyperplane $\left(a^i x = a_0^i\right)$, it is intuitive that not all the coefficients μ_j^i in the representation (4.43) can be close to 0 or 1. We formally prove this observation in the next claim.

Claim 4.15 *Let $K > 1$ be a constant. There exists an integer $N_1 = N_1(K)$ such that for every $i \geq N_1$, there exists an index $j \in \{1, \ldots, l\}$ such that the coefficient μ_j^i in* (4.43) *satisfies*

$$\frac{K}{q_j^i} \leq \mu_j^i \leq 1 - \frac{K}{q_j^i} \ .$$

Proof of claim. By Claim 4.14, any vertex f^i of the parallelepiped

$$z_0^i + \bar{\Pi}(u_1, \ldots, u_k, v_1^i, \ldots, v_l^i)$$

satisfies $\|x_0 - f^i\| \longrightarrow \infty$. Therefore,

$$\left\|x_0 - f^i\right\| \leq \left\|x_0 - \tilde{x}_0^i\right\| + \left\|\tilde{x}_0^i - f^i\right\| \longrightarrow \infty \ .$$

Because (4.42) implies $\|x_0 - \tilde{x}_0^i\| \longrightarrow 0$, we must have

$$\left\|\tilde{x}_0^i - f^i\right\| \longrightarrow \infty \ .$$

In particular, there exists a number N_1 such that for all $i \geq N_1$

$$\left\|\tilde{x}_0^i - f^i\right\| > \sum_{p=1}^{k} \|u_p\| + 2KRl \ .$$

Now let $i \geq N_1$ and assume that there are index sets J_1^i and J_2^i such that $J_1^i \cup J_2^i = \{1, \ldots, l\}$ and such that for every index $j \in J_1^i$, we have $0 \leq \mu_j^i < K/q_j^i$, and for every index $j \in J_2^i$, it holds that $0 \leq 1 - \mu_j^i < K/q_j^i$. For the vertex

$$f^i = z_0^i + \sum_{j \in J_2^i} v_j^i \ ,$$

of the parallelepiped it follows with property (vi) from Lemma 4.8 that

$$\begin{aligned}
\left\|\tilde{x}_0^i - f^i\right\| &= \left\|\sum_{p=1}^{k} \gamma_p^i u_p + \mu_1^i \delta_1^i v_1^i + \ldots + \mu_l^i \delta_l^i v_l^i - \sum_{j \in J_2^i} \delta_j^i v_j^i\right\| \\
&\leq \sum_{p=1}^{k} \|u_p\| + \left\|\sum_{j \in J_1^i} \mu_j^i \delta_j^i v_j^i - \sum_{j \in J_2^i} (1-\mu_j^i) \delta_j^i v_j^i\right\| \\
&\leq \sum_{p=1}^{k} \|u_p\| + \sum_{j \in J_1^i} \mu_j^i \left\|v_j^i\right\| + \sum_{j \in J_2^i} (1-\mu_j^i) \left\|v_j^i\right\| \\
&\leq \sum_{p=1}^{k} \|u_p\| + K \sum_{j=1}^{l} \frac{\left\|v_j^i\right\|}{q_j^i} \longrightarrow \sum_{p=1}^{k} \|u_p\| + KRl \ ,
\end{aligned}$$

which is a contradiction. ☐

The next technical claim is needed to choose a proper parameter α for the vectors $h_{\alpha^i}^i(\delta)$ in Lemma 4.8 that give rise to appropriate Gomory-Chvátal cuts.

Claim 4.16 *Let $K > 1$, $\mu \in [0,1]$ and $q \in \mathbb{R}$ such that $q \geq 2K$ and $K/q \leq \mu \leq 1 - K/q$. Then there exist integers p_1 and p_2 such that $1 \leq p_1 \leq q/(2K)$ and*

$$p_2 + 1/4 \leq \mu p_1 \leq (p_2 + 1) - 1/4 \ .$$

Proof of claim. We consider three cases. If $1/4 \leq \mu \leq 1 - 1/4$, then $p_1 = 1$ and $p_2 = 0$ satisfy the conditions of the claim. If $\mu < 1/4$, there must exist an integer p such that $1/4 \leq \mu p \leq 1/2 \leq 1 - 1/4$. Then

$$1 \leq \frac{1}{4\mu} \leq p \leq \frac{1}{2\mu} \leq \frac{q}{2K} \ ,$$

and we can set $p_1 = p$ and $p_2 = 0$. Finally, if $\mu > 1 - 1/4$, then $1 - \mu < 1/4$ and there must exist an integer p such that $1/4 \leq (1-\mu)p \leq 1/2$. Then

$$1 \leq \frac{1}{4(1-\mu)} \leq p \leq \frac{1}{2(1-\mu)} \leq \frac{q}{2K} \ .$$

For $p_1 = p$ and $p_2 = p - 1$, we get

$$p_2 + 1/4 < p - 1/2 \leq \mu p \leq p - 1/4 = (p_2 + 1) - 1/4 \ .$$

$\square$

For the remainder, let us fix a constant K such that $K > 8(2+l)$. For large enough i, the assumptions of Claim 4.16 are satisfied, that is, $q^i_j \geq 2K$ for every $j = 1, \ldots, l$. Then Claims 4.15 and 4.16 imply that there exists an integer $N(K)$ such that for every $i \geq N(K)$, there exists an index $s \in \{1, \ldots, l\}$ and integer numbers p^i_1 and p^i_2 such that

$$1 \leq p^i_1 \leq \frac{q^i_s}{2K}$$

and

$$p^i_2 + \frac{1}{4} \leq \mu^i_s p^i_1 \leq (p^i_2 + 1) - \frac{1}{4} \ .$$

Note that we can write the positive integer p^i_1 as $\lfloor \bar{\alpha}^i_s q^i_s \rfloor$ for some scalar $\bar{\alpha}^i_s$. That is, there exist a number $0 < \bar{\alpha}^i_s < 1/K$ and an integer p^i such that

$$p^i + 1/4 \leq \mu^i_s \left\lfloor \bar{\alpha}^i_s q^i_s \right\rfloor \leq (p^i + 1) - 1/4 \ . \tag{4.44}$$

Define $\alpha^i \in \mathbb{R}^l_+$ by

$$\alpha^i_j = \begin{cases} \bar{\alpha}^i_s, & \text{if } j = s \\ 0, & \text{otherwise} \ . \end{cases}$$

Note that $\|\alpha^i\|_\infty \leq 1$. Now take $h^i := h^i_{\alpha^i}(\bar{\delta})$ according to Lemma 4.8 from property (x) for $\bar{\delta} = (1, \ldots, 1)$. For some sufficiently large number N_2, we can assume that for every $i \geq N_2$,

$$h^i \perp u_p \qquad \text{for } p = 1, \ldots, k \tag{4.45}$$

$$\alpha^i_j - 1/K \leq w^i_j \, h^i \leq \alpha^i_j + 1/K \qquad \text{for } j = 1, \ldots, l \tag{4.46}$$

$$h^i \, v^i_j = \left\lfloor \alpha^i_j q^i_j \right\rfloor \qquad \text{for } j = 1, \ldots, l \tag{4.47}$$

$$0 \geq \, r_s \, h^i \qquad \text{for } s = 1, \ldots, m \tag{4.48}$$

$$\left| h^i \, a^i \right| \leq C \left\| a^i \right\|^2 \ , \tag{4.49}$$

where $C > 0$ is a constant. By (4.40) and arguing as in part (b), R has been chosen large enough so that every point $x \in F$ can be written as

$$x = x_0 + u + \sum_{j=1}^{l} \lambda_j \delta_j w_j^i \ ,$$

for some $u \in U$ and $\lambda_j \in [0,1]$ and $\delta_j \in \{-1,1\}$ for $j = 1,\ldots,l$. With (4.41), (4.43), (4.45) and (4.47), we get for every $x \in F$,

$$\begin{aligned}
h^i x &= h^i x_0 + h^i u + \sum_{j=1}^{l} \lambda_j h^i \delta_j w_j^i = h^i x_0 + \sum_{j=1}^{l} \lambda_j h^i \delta_j w_j^i \\
&= h^i z_0^i + \mu_1^i \, h^i \, v_1^i + \ldots + \mu_l^i \, h^i \, v_l^i - \frac{a_0^i - a^i x_0}{\|a^i\|^2} \, h^i a^i + \sum_{j=1}^{l} \lambda_j h^i \delta_j w_j^i \\
&= h^i z_0^i + \mu_1^i \left\lfloor \alpha_1^i q_1^i \right\rfloor + \ldots + \mu_l^i \left\lfloor \alpha_l^i q_l^i \right\rfloor - \frac{a_0^i - a^i x_0}{\|a^i\|^2} \, h^i a^i + \sum_{j=1}^{l} \lambda_j h^i \delta_j w_j^i \\
&= h^i z_0^i + \mu_s^i \left\lfloor \bar{\alpha}_s^i q_s^i \right\rfloor - \frac{a_0^i - a^i x_0}{\|a^i\|^2} \, h^i a^i + \sum_{j=1}^{l} \lambda_j h^i \delta_j w_j^i \ .
\end{aligned}$$

For large enough i we get with (4.42) and (4.49),

$$\left| \frac{a_0^i - a^i x_0}{\|a^i\|^2} \, h^i a^i \right| \leq 1/K \ .$$

Consequently, with (4.46) and $0 \leq \bar{\alpha}_s^i < 1/K$, we obtain

$$\begin{aligned}
\left| -\frac{a_0^i - a^i x_0}{\|a^i\|^2} \, h^i a^i + \sum_{j=1}^{l} \lambda_j h^i \delta_j w_j \right| &\leq \left| \frac{a_0^i - a^i x_0}{\|a^i\|^2} \, h^i a^i \right| + \sum_{j=1}^{l} \left| h^i w_j \right| \\
&\leq \frac{1}{K} + \bar{\alpha}_s^i + \frac{l}{K} \leq \frac{2+l}{K} < \frac{1}{8} \ .
\end{aligned}$$

This implies that for every $x \in F$,

$$h^i z_0^i + \mu_s^i \left\lfloor \bar{\alpha}_s^i q_s^i \right\rfloor - \frac{1}{8} \leq h^i x \leq h^i z_0^i + \mu_s^i \left\lfloor \bar{\alpha}_s^i q_s^i \right\rfloor + \frac{1}{8} \ ,$$

and with (4.44), it follows that for every $x \in F$,

$$(h^i z_0^i + p^i) + 1/8 \leq h^i x \leq (h^i z_0^i + p^i + 1) - 1/8 \ . \tag{4.50}$$

Now observe that (4.48) implies that

$$\max\left\{h^i x \mid x \in P\right\} = \max\left\{h^i x \mid x \in F\right\} \ .$$

Therefore, using (4.50) and the fact that $z_0^i \in \mathbb{Z}^n$, we have that $h^i x \leq h^i z_0^i + p^i$ is a Gomory-Chvátal cut for P. Moreover, (4.50) implies that this cut is violated by every point in F, that is,

$$\left(h^i x \leq h^i z_0^i + p^i\right) \cap F = \emptyset \ .$$

Arguing as at the beginning of part (c), we can find a rational polyhedron $Q \supseteq P$ such that $(h^i x \leq h^i z_0^i + p^i)$ is also a Gomory-Chvátal cut for Q and such that

$$Q \cap \left(h^i x \leq h^i z_0^i + p^i\right) \subseteq (ax \leq a_P) \ .$$

The facet normals of Q together with h^i imply the desired set S of the lemma. □

As the proof of the above lemma shows, for every non-rational face-defining inequality $ax \leq a_P$ of P, the Gomory-Chvátal procedure will separate every point in $P \cap (ax = a_P)$ that is not contained in the maximal rational affine subspace of $(ax = a_P)$.

Corollary 4.17 *Let P be a polytope and let $F = P \cap (ax = a_P)$ be a face of P. If V_R denotes the maximal rational affine subspace of $(ax = a_P)$, then $P' \cap F \subseteq V_R$.*

Lemma 4.13 gives us the tools to complete the first step of the main proof.

Corollary 4.18 *Let P be a polytope in $\mathbb{R}^n$. Then there exists a finite set $S \subseteq \mathbb{Z}^n$ such that $C_S(P) \subseteq P$.*

Proof. Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ for some matrix A and some vector b. Let A^1 denote the set of vectors corresponding to rows of A that define rational facet-defining inequalities of P and let A^2 denote the set of vectors associated with the non-rational facet-defining inequalities of P. By means of Lemma 4.13, for every non-rational facet-defining inequality $ax \leq a_P$ of P, there exists a finite set $S_a \subseteq \mathbb{Z}^n$ such that $C_{S_a}(P) \subseteq (ax \leq a_P)$. Therefore, the finite set

$$S := \left(\bigcup_{a \in A^2} S_a \right) \cup A^1$$

satisfies $C_S(P) \subseteq P$. □

4.4.3 Step 2

In this part of the proof, we establish a connection between the Gomory-Chvátal closure of a face F of a polytope P (seen as a lower-dimensional polytope in $\mathbb{R}^n$) and the Gomory-Chvátal closure of the polytope itself. In particular, we show that F behaves under its closure F' as it behaves under the closure P', in short, $F' = P' \cap F$. This relationship has long been established for rational polytopes and can, in this case, be immediately derived from the following lemma (see Cook, Cunningham, Pulleyblank, and Schrijver 1998, Lemma 6.33).

Lemma 4.19 *Let P be a rational polytope and let $F = P \cap (ax = a_P)$ be a face of P. If $cx \leq \lfloor c_F \rfloor$ is a Gomory-Chvátal cut for F, then there exists a Gomory-Chvátal cut $\bar{c}x \leq \lfloor \bar{c}_P \rfloor$ for P such that*

$$(ax = a_P) \cap (cx \leq \lfloor c_F \rfloor) = (ax = a_P) \cap (\bar{c}x \leq \lfloor \bar{c}_P \rfloor) \ .$$

Geometrically, this lemma says that every Gomory-Chvátal cut for F can be "rotated" so that it becomes a Gomory-Chvátal cut for P that has the same effect on the hyperplane defining F.

Corollary 4.20 *Let F be a face of a rational polyhedron P. Then*

$$F' = P' \cap F \ .$$

This last observation was the main ingredient to Schrijver's result that every rational polyhedron has a finite Chvátal rank (Schrijver 1980), and it has also been useful to Bockmayr, Eisenbrand, Hartmann, and Schulz (1999) who proved that the Chvátal rank of polytopes in the unit cube is polynomially bounded in the dimension. Both proofs use induction on the dimension of the polyhedron and induction is also our motivation to extend Corollary 4.20 to arbitrary polytopes. If we were able to show a similar rotation property as in Lemma 4.19 for non-rational faces, which would imply that $F' = P' \cap F$ also holds for non-rational polytopes, an inductive argument for our main result might be as follows: Knowing that the Gomory-Chvátal closure of a lower-dimensional facet F of P is a polytope, we would rotate each of the finite number of cuts that describe F' in order to obtain Gomory-Chvátal cuts for P that play the same role for $P' \cap F$. In doing so, we would establish the existence of a finite set $S_F \subseteq \mathbb{Z}^n$ with the property that $C_{S_F}(P) \cap F = F'$. As the facets of P constitute the relative boundary of the polytope, the union of all these sets would satisfy property (P2) in Figure 4-3.

We cannot hope that Lemma 4.19 holds in the exact same way for non-rational polytopes. For this, we had to rotate the valid cut for F with the non-rational vector a of the non-rational facet-defining inequality. But such rotation would not result in a *rational* inequality $\bar{c}x \leq \bar{c}_P$ for P and it would, therefore, not correspond to a Gomory-Chvátal cut. However, we can show a weaker statement: For every undominated Gomory-Chvátal cut of the face $F = P \cap (ax = a_P)$, one can find a Gomory-Chvátal cut for P that has the same impact on the maximal rational affine subspace V_R of $(ax = a_P)$ as the cut for F. As we showed in Step 1 of the proof, every point in $F \setminus V_R$ is separated by some Gomory-Chvátal cut (see also Corollary 4.17). As a consequence, this weaker property would still suffice to show that $F' = P' \cap F$ applies to arbitrary polytopes.

Lemma 4.21 *Let P be a polytope and let $F = P \cap (ax = a_P)$ be a face of P such that $(ax = a_P)$ is a non-rational hyperplane. Let V_R denote the maximal rational affine subspace of $(ax = a_P)$ and assume that $V_R \neq \emptyset$. If $cx \leq \lfloor c_F \rfloor$ is a Gomory-Chvátal cut for F and facet-defining for F', then there exists a Gomory-Chvátal cut $\bar{c}x \leq \lfloor \bar{c}_P \rfloor$ for P such that*

$$(ax = a_P) \cap V_R \cap (\bar{c}x \leq \lfloor \bar{c}_P \rfloor) = (ax = a_P) \cap V_R \cap (cx \leq \lfloor c_F \rfloor) \ .$$

Proof. Let $V_R = x_0 + \mathrm{span}(u_1, \ldots, u_k)$, where $x_0 \in (ax = a_P) \cap \mathbb{Q}^n$ and $u_1, \ldots, u_k \in \mathbb{Z}^n$. Note that $V_R = \{x_0\}$ is possible. Furthermore, assume that $P \subseteq (ax \leq a_P)$. Now consider a Gomory-Chvátal cut $cx \leq \lfloor c_F \rfloor$ for F that is facet-defining for F'. Moreover, assume that $\hat{x}$ is a vertex of F that maximizes c over F. Let $r_1, \ldots, r_m$ denote all edge directions of P that emanate from vertices in F to vertices of P that are not in F. Note that for $s = 1, \ldots, m$,

$$r_s\, a < 0 \ . \tag{4.51}$$

According to Corollary 4.4, there exists a sequence $\{a^i\} \subseteq \mathbb{Z}^n$ such that $a^i \perp u_j$, for $j = 1, \ldots, k$, and such that

$$\left\|a^i\right\| \left\|\bar{a}^i - \bar{a}\right\| \ \longrightarrow \ 0 \ , \tag{4.52}$$

where $\bar{a}^i = a^i / \left\|a^i\right\|$ and $\bar{a} = a / \|a\|$. As $r_s \bar{a} < 0$ by (4.51), it follows with (4.52) that there exists a constant $\beta > 0$ such that $r_s \bar{a}^i \leq -\beta$ for large enough i. Hence, noting that $\|a^i\| \longrightarrow \infty$ because of $a \in \mathbb{R}^n \backslash \mathbb{Q}^n$, there exists a constant $\beta > 0$ and an $N_1 \in \mathbb{N}$ such that $r_s a^i \leq -\beta$ for all $s = 1, \ldots, m$ and $i \geq N_1$. Let $M := \max_{s=1,\ldots,m}\{c\, r_s\}$. If $M \leq 0$, then $\hat{x}$ also maximizes c over P and, hence, $cx \leq \lfloor c_F \rfloor$ is a Gomory-Chvátal cut for P. Therefore, assume that $M > 0$. Let $p \in \mathbb{Z}^n$ and $q \in \mathbb{N}$ with $q \geq 1$ such that $x_0 = p/q$. We define the constant $K := q\lceil \frac{1}{\beta} M \rceil$ and vectors $\vec{c}^{\,i} := c + K\, a^i$ for every $i \geq N_1$. Note that $K \in \mathbb{Z}$ and therefore $\vec{c}^{\,i} \in \mathbb{Z}^n$. We have for $s = 1, \ldots, m$,

$$r_s \vec{c}^{\,i} = r_s(c + K\, a^i) \leq r_s c - K\beta \leq 0 \ ,$$

which implies that for $i \geq N_1$, the vector $\vec{c}^{\,i}$ is maximized over P by a point in F. Now let $\hat{x}^i \in \arg\max\{a^i x \mid x \in F\}$. We obtain for every $i \geq N_1$,

$$\begin{aligned}
\max\left\{\vec{c}^{\,i} x \mid x \in P\right\} &= \max\left\{\vec{c}^{\,i} x \mid x \in F\right\} \\
&\leq \max\{cx \mid x \in F\} + K \max\left\{a^i x \mid x \in F\right\} \\
&= c\hat{x} + K a^i \hat{x}^i \\
&= c_F + K a^i x_0 + K a^i (\hat{x}^i - x_0) \ .
\end{aligned}$$

With (4.52), the boundedness of F, and $a\hat{x}^i = ax_0 = a_P$, we get

$$\left|a^i(\hat{x}^i - x_0)\right| = \left\|a^i\right\| \left|(\bar{a}^i - \bar{a})(\hat{x}^i - x_0)\right| \leq \left\|a^i\right\| \left\|\bar{a}^i - \bar{a}\right\| \left\|\hat{x}^i - x_0\right\| \longrightarrow 0 \ .$$

Therefore, for any $\varepsilon > 0$, there exists an $N_\varepsilon \in \mathbb{N}$ such that $|Ka^i(\hat{x}^i - x_0)| \leq \varepsilon$ for all $i \geq N_\varepsilon$. In particular, we can choose i large enough so that

$$\bar{c}^i_P = \max\left\{\bar{c}^i x \mid x \in P\right\} < \lfloor c_F \rfloor + Ka^i x_0 + 1 \ .$$

Observe that $Ka^i x_0 \in \mathbb{Z}$. Consequently,

$$\bar{c}^i\, x \leq \left\lfloor \bar{c}^i_P \right\rfloor \leq \lfloor c_F \rfloor + Ka^i x_0 \tag{4.53}$$

is a Gomory-Chvátal cut for P. Furthermore, it has to hold that $\lfloor \bar{c}^i_P \rfloor = \lfloor c_F \rfloor + Ka^i x_0$: First, observe that Corollary 4.18 implies that $F' \subseteq F$ and, therefore, $F' \subseteq P' \cap F$. As $cx \leq \lfloor c_F \rfloor$ is by assumption facet-defining for F', there must exist a point $\tilde{x} \in F'$ such that $c\tilde{x} = \lfloor c_F \rfloor$. Note that $F' \subseteq V_R$, according to Corollary 4.17, implies that $\tilde{x} \in V_R$ and, thus, $a^i\tilde{x} = a^i x_0$. Furthermore, we have $\tilde{x} \in P' \cap F$ because of $F' \subseteq P' \cap F$. In particular, $\tilde{x}$ satisfies the Gomory-Chvátal cut $\bar{c}^i x \leq \lfloor \bar{c}^i_P \rfloor$. Consequently,

$$\bar{c}^i \tilde{x} = c\tilde{x} + Ka^i\tilde{x} = \lfloor c_F \rfloor + Ka^i x_0 \leq \left\lfloor \bar{c}^i_P \right\rfloor \ .$$

Together with (4.53), we obtain $\lfloor \bar{c}^i_P \rfloor = \lfloor c_F \rfloor + Ka^i x_0$. It follows that

$$\begin{aligned}
\left(\bar{c}^i\, x \leq \left\lfloor \bar{c}^i_P \right\rfloor\right) \cap \left(a^i x = a^i x_0\right) &= \left(cx + Ka^i x \leq \lfloor c_F \rfloor + Ka^i x_0\right) \cap \left(a^i x = a^i x_0\right) \\
&= \left(cx \leq \lfloor c_F \rfloor\right) \cap \left(a^i x = a^i x_0\right) \ .
\end{aligned}$$

As $V_R \subseteq (a^i x = a^i x_0)$, this implies for $\bar{c} := \bar{c}^i$ for some large enough i,

$$\left(\bar{c}\, x \leq \lfloor \bar{c}_P \rfloor\right) \cap V_R = \left(cx \leq \lfloor c_F \rfloor\right) \cap V_R \ .$$

The lemma follows. $\square$

With this observation, we can prove the homogeneity property for non-rational polytopes.

Corollary 4.22 *Let P be a polytope and let F be a face of P. Then $F' = P' \cap F$.*

Proof. For the first direction $F' \subseteq P' \cap F$, observe that $F \subseteq P$ implies $F' \subseteq P'$. Furthermore, $F' \subseteq F$ because of Corollary 4.18. Hence, $F' \subseteq P' \cap F$.

For the second direction, let $F = P \cap (ax = a_P)$ be a face of P and let $cx \leq \lfloor c_F \rfloor$ be a Gomory-Chvátal cut for F that is facet-defining for F'. If $(ax = a_P)$ is a rational hyperplane, then we can find a rational polytope $Q \supseteq P$ such that $F_Q = Q \cap (ax = a_P)$ is a face of Q and such that $cx \leq \lfloor c_F \rfloor$ is a Gomory-Chvátal cut for F_Q. By Lemma 4.19, there exists a Gomory-Chvátal cut $\bar{c}\,x \leq \lfloor \bar{c}_Q \rfloor$ for Q such that

$$\big(c\,x \leq \lfloor c_F \rfloor\,\big) \cap (ax = a_P) = \big(\bar{c}\,x \leq \lfloor \bar{c}_Q \rfloor\,\big) \cap (ax = a_P) \ .$$

Then $P \subseteq Q$ implies that $\bar{c}\,x \leq \lfloor \bar{c}_Q \rfloor$ is also a Gomory-Chvátal cut for P and we have $P' \cap F \subseteq (cx \leq \lfloor c_F \rfloor)$. Now assume that $(ax = a_P)$ is non-rational. If $(ax = a_P) \cap \mathbb{Q}^n = \emptyset$, then Corollary 4.17 implies $P' \cap F = \emptyset \subseteq F'$. Therefore, assume that $(ax = a_P) \cap \mathbb{Q}^n \neq \emptyset$, that is, the maximal rational affine subspace V_R of $(ax = a_P)$ is non-empty. By Lemma 4.21, there exists a Gomory-Chvátal cut for P that satisfies

$$(ax = a_P) \cap V_R \cap \big(\bar{c}x \leq \lfloor \bar{c}_P \rfloor\,\big) = (ax = a_P) \cap V_R \cap \big(cx \leq \lfloor c_F \rfloor\,\big) \ .$$

Together with Corollary 4.17, that is, $P' \cap F \subseteq V_R$, we obtain

$$P' \cap F \subseteq \big(cx \leq \lfloor c_F \rfloor\,\big) \ .$$

□

4.4.4 Step 3

In this subsection, we show Part II of our general proof strategy outlined in Figure 4-3: If for some finite set $S \subseteq \mathbb{Z}^n$ of vectors

$$C_S(P) \subseteq P \qquad \text{and} \qquad C_S(P) \cap \mathrm{rbd}(P) \subseteq P' \ ,$$

not more than a finite number of Gomory-Chvátal cuts have to be added to $C_S(P)$ to obtain the Gomory-Chvátal closure P'. In fact, we prove that this is true for arbitrary bounded convex set.

Dadush, Dey, and Vielma (2010) proved this property for full-dimensional convex sets K. The key observation in this case was that one can find an ε-ball around every interior point of K such that the ball is contained in K. Since any additional undominated cut for K' must separate a vertex of $C_S(K)$ in the strict interior of K, it must be derived from an inequality for which the boundary of the associated half-space is shifted by at least ε. However, for valid inequalities $ax \leq a_P$ with $\|a\| > 1/\varepsilon$ this is not possible (see also Figure 4-2). As a consequence, only cuts that are associated with normal vectors of a certain bounded norm need to be considered and their number is finite.

For a lower-dimensional convex set K the situation is a somewhat different. Any additional undominated cut would have to separate a point v in the *relative* interior of K and no ε-ball around v is fully contained in K. All we can guarantee is that there exists an ε-ball whose intersection with the affine hull of K and, hence, also with the affine hull of $C_S(K)$, is contained in K. Therefore, any cut that separates v has to correspond to a half-space $(cx \leq c_K)$ for which the intersection of its boundary $(cx = c_K)$ with the affine hull $\operatorname{aff}(C_S(K))$ is shifted by at least ε within $\operatorname{aff}(C_S(K))$. This intersection of $(cx = c_K)$ with $\operatorname{aff}(C_S(K))$ is a lower-dimensional rational affine subspace, say H^*. Similar to the full-dimensional case, there is only a finite number of these affine subspaces H^*, which are shifted within $\operatorname{aff}(C_S(K))$ by at least a distance of ε by the rounding operation in the Gomory-Chvátal procedure. However, an infinite number of hyperplanes in $\mathbb{R}^n$ has the same intersection H^* with $\operatorname{aff}(C_S(K))$. Consequently, there is an infinite number of Gomory-Chvátal cuts that could separate a point in the relative interior of $C_S(K)$. Yet, we show that among all rational valid inequalities $cx \leq c_K$ with the same intersection $(cx = c_K) \cap \operatorname{aff}(C_S(K))$, there will be one that corresponds to a Gomory-Chvátal cut that dominates every other Gomory-Chvátal cut associated with a valid inequality in this "equivalence class". For this reason, no more than one Gomory-Chvátal cut for each of the finitely many H^*'s has to be added to the description of $C_S(K)$.

In the following lemma, we formalize this observation and prove the finite augmentation property for arbitrary bounded convex sets.

Lemma 4.23 *Let K be a convex and compact set in $\mathbb{R}^n$. If there exists a finite set $S \subseteq \mathbb{Z}^n$ such that*

(i) $C_S(K) \subseteq K$,

(ii) $C_S(K) \cap \operatorname{rbd}(K) \subseteq K'$,

then K' is a rational polytope.

Proof. As $C_S(K)$ is a rational polytope, we can assume that $\operatorname{aff}(C_S(K)) = w_0 + W$, where $w_0 \in \mathbb{Q}^n$ and where W is a rational linear vector space. Let $\mathcal{V}$ denote the finite set of vertices of $C_S(K)$. Because of assumption (ii), any Gomory-Chvátal cut for K that separates a point in $C_S(K) \setminus K'$ must also separate a vertex in $\mathcal{V} \setminus \operatorname{rbd}(K)$. We will show that for each of the finitely many vertices of $C_S(K)$ in the relative interior of K one only has to consider a finite set of Gomory-Chvátal cuts.

First, observe that because of $\mathcal{V} \setminus \operatorname{rbd}(K) \subseteq \operatorname{ri}(K)$ and since the number of vertices of $C_S(K)$ is finite, there exists an $\varepsilon > 0$ such that for every $v \in \mathcal{V} \setminus \operatorname{rbd}(K)$,

$$\big(v + B(0,\varepsilon)\big) \cap \operatorname{aff}(K) \subseteq K \ . \tag{4.54}$$

Consequently,

$$\big(v + B(0,\varepsilon)\big) \cap \operatorname{aff}\big(C_S(K)\big) \subseteq K \ . \tag{4.55}$$

Now let us fix a vertex v of $C_S(K)$ in the relative interior of K, that is, $v \in \mathcal{V} \setminus \operatorname{rbd}(K)$. Furthermore, let $c \in \mathbb{Z}^n$. We will consider two cases, depending on whether K is full-dimensional or not. If $\dim(K) = n$, then $\operatorname{aff}(K) = \mathbb{R}^n$ and with (4.54),

$$\big(v + B(0,\varepsilon)\big) \subseteq K \ .$$

We get

$$\begin{aligned} \lfloor c_K \rfloor &= \lfloor \max\{cx \mid x \in K\} \rfloor \geq \max\{cx \mid x \in K\} - 1 \\ &\geq c\,v + \max\big\{cx \mid x \in B(0,\varepsilon)\big\} - 1 = c\,v + c\left(\varepsilon \frac{c}{\|c\|}\right) - 1 = c\,v + \varepsilon\,\|c\| - 1 \ . \end{aligned}$$

If $\|c\| \geq 1/\varepsilon$, then the Gomory-Chvátal cut associated with the normal vector c does not separate the vertex v. Hence, we only need to consider Gomory-Chvátal cuts that correspond to vectors c such that $\|c\| < 1/\varepsilon$ and their number is finite. This completes the proof for the case that K is full-dimensional.

In the remainder of the proof, let us assume that $\dim(K) < n$ and, therefore, $\dim(\mathrm{aff}(C_S(K)) =: k < n$. Since $\dim(W) = k$, we can rename the indices such that there exist integers p_{ij} and $q_{ij} \geq 1$, for $i = 1, \ldots, n-k$ and $j = 1, \ldots, k$, such that for every $x \in W$,

$$x_{k+i} = \sum_{j=1}^{k} \frac{p_{ij}}{q_{ij}} x_j \enspace .$$

In words, any point in W is uniquely determined by its first k components. Moreover, we can find an upper bound for the norm of each point $x \in W$ that is a function of the norm of the vector $(x_1, \ldots, x_k)$, that is, the restriction of x to its first k components: Since

$$\|x\|^2 = x_1^2 + \ldots + x_k^2 + \left(\sum_{j=1}^{k} \frac{p_{1j}}{q_{1j}} x_j \right)^2 + \ldots + \left(\sum_{j=1}^{k} \frac{p_{n-k,j}}{q_{n-k,j}} x_j \right)^2 \enspace ,$$

there exist rational constants $\alpha_i > 0$, for $i = 1, \ldots, k$, and α_{ij}, for $1 \leq i < j \leq k$, such that

$$\|x\|^2 = \alpha_1 x_1^2 + \ldots + \alpha_k x_k^2 + \sum_{1 \leq i < j \leq k} \alpha_{ij} x_i x_j \enspace .$$

Using the relation

$$\alpha_{ij}\, x_i\, x_j \leq \frac{1}{2} |\alpha_{ij}|\, x_i^2 + \frac{1}{2} |\alpha_{ij}|\, x_j^2 \enspace ,$$

and defining $\alpha_{ji} := \alpha_{ij}$ for every $1 \leq i < j \leq k$, we obtain

$$\|x\|^2 \leq \left(\alpha_1 + \frac{1}{2} \sum_{j=2}^{k} |\alpha_{1j}| \right) x_1^2 + \ldots + \left(\alpha_k + \frac{1}{2} \sum_{j=1}^{k-1} |\alpha_{kj}| \right) x_k^2 \enspace .$$

Let $\alpha := \max_{i=1,\ldots,k} \left\{ \alpha_i + \frac{1}{2} \sum_{j=1, j \neq i}^{k} |\alpha_{ij}| \right\}$. Then α is a constant that only depends on W. For any $x \in W$, we have

$$\|x\| \leq \sqrt{\alpha}\, \|(x_1, \ldots, x_k)\| \enspace .$$

Moreover,

$$c\,x \;=\; c_1x_1+\ldots+c_kx_k+\sum_{i=1}^{n-k}c_{k+i}\left(\sum_{j=1}^{k}\frac{p_{ij}}{q_{ij}}\,x_j\right)=\sum_{j=1}^{k}\left(c_j+\sum_{i=1}^{n-k}\frac{p_{ij}}{q_{ij}}\,c_{k+i}\right)x_j\ .$$

Let $L:\mathbb{R}^n\to\mathbb{R}^k$ denote the affine map that is defined for $j=1,\ldots,k$ by

$$L_j(x):=x_j+\sum_{i=1}^{n-k}\frac{p_{ij}}{q_{ij}}\,x_{k+i}\ . \tag{4.56}$$

Then for every $x\in W$,

$$cx=\sum_{j=1}^{k}L_j(c)\,x_j=L(c)\,(x_1,\ldots,x_k)\ . \tag{4.57}$$

Let $x^c=(x_1^c,\ldots,x_n^c)\in W$ such that $(x_1^c,\ldots,x_k^c)=L(c)$. Observe that

$$\|x^c\|\le\sqrt{\alpha}\ \|(x_1^c,\ldots,x_k^c)\|=\sqrt{\alpha}\ \|L(c)\|$$

implies

$$\frac{1}{\sqrt{\alpha}\,\|L(c)\|}\,x^c\in B(0,1)\cap W\ .$$

Using $\mathrm{aff}(C_S(K))=v+W$, we get

$$v+\frac{\varepsilon}{\sqrt{\alpha}\,\|L(c)\|}x^c\in\Big(\big(v+B(0,\varepsilon)\big)\cap\mathrm{aff}\big(C_S(K)\big)\Big)\ ,$$

and by (4.55),

$$v+\frac{\varepsilon}{\sqrt{\alpha}\,\|L(c)\|}x^c\in K\ .$$

Therefore, and with (4.57), we get

$$\begin{aligned}\lfloor c_K\rfloor &= \lfloor\max\{cx\mid x\in K\}\rfloor\ge\max\{cx\mid x\in K\}-1\\[2ex] &\ge c\,v+\frac{\varepsilon}{\sqrt{\alpha}\,\|L(c)\|}\,c\,x^c-1=c\,v+\frac{\varepsilon}{\sqrt{\alpha}}\,\|L(c)\|-1\ .\end{aligned}$$

For $\|L(c)\| \geq \sqrt{\alpha}/\varepsilon$, the Gomory-Chvátal cut associated with c does not separate v. Because of (4.56), for each $j = 1, \ldots, k$, there exists an integer $q_j \geq 1$ such that $L_j(c)$ is a multiple of $1/q_j$. Therefore, the number of vectors $L(c)$ in $\mathbb{R}^k$ with $\|L(c)\| < \sqrt{\alpha}/\varepsilon$ is finite. However, there is an infinite number of integral vectors c in $\mathbb{R}^n$ that are mapped to the same rational vector $L(c)$ in $\mathbb{R}^k$. Let $\mathcal{A}$ denote the set of rational vectors $a \in \mathbb{R}^k$ such that a_j is a multiple of q_j, for $j = 1, \ldots, k$, and such that $\|a\| < \sqrt{\alpha}/\varepsilon$. For every $a \in \mathcal{A}$, we define

$$N(a) := \{c \in \mathbb{Z}^n \mid L(c) = a\} \ .$$

Let

$$c^a \in \arg \min_{c \in N(a)} \left\{ \lfloor c_K \rfloor - cv \right\} \ .$$

Observe, that c^a is well-defined: Since $v \in K$, we have for any $c \in N(a)$,

$$\lfloor c_K \rfloor - cv \geq \max\{cx \mid x \in K\} - 1 - cv \geq -1 \ .$$

Furthermore, as v is a vertex of the rational polytope $C_S(K)$, it holds that $v \in \mathbb{Q}^n$. Hence, there exist an integer vector $\bar{v} \in \mathbb{Z}^n$ and an integer $q_v \geq 1$ such that $v = \bar{v}/q_v$. Consequently, the set $\left\{ \lfloor c_K \rfloor - cv \mid c \in N(a) \right\}$ contains only multiples of q_v and is bounded from below.

Finally, observe that the Gomory-Chvátal cut $c^a\, x \leq \lfloor c^a_K \rfloor$ dominates every other Gomory-Chvátal cut associated with a vector in $N(a)$ in $\operatorname{aff}(C_S(K))$: For this, consider an arbitrary point $x \in \operatorname{aff}(C_S(K))$. We can write $x = v + w$, for some $w = (w_1, \ldots, w_n) \in W$. If, using (4.57),

$$c^a x = c^a v + c^a w = c^a v + L(c^a)(w_1, \ldots, w_k) = c^a v + a(w_1, \ldots, w_k) \leq \lfloor c^a_K \rfloor \ \ ,$$

then

$$a(w_1, \ldots, w_k) \leq \lfloor c^a_K \rfloor - c^a v \ .$$

By the definition of c^a, it follows that for every $c \in N(a)$,

$$cx = cv + cw = cv + a(w_1, \ldots, w_k) \leq cv + \lfloor c_K \rfloor - cv = \lfloor c_K \rfloor \ \ .$$

That is, if x satisfies the Gomory-Chvátal cut $c^a\, x \leq \lfloor c^a_K \rfloor$, then it also satisfies every

Gomory-Chvátal cut $cx \leq \lfloor c_K \rfloor$ with $c \in N(a)$. Consequently, for each vector $a \in \mathcal{A}$, we only need to consider a single Gomory-Chvátal cut. Since $|\mathcal{A}|$ is finite, this completes the proof. $\square$

4.4.5 Step 4

We are finally prepared to prove the main result of this chapter. By drawing from the insights of the previous three subsections and using an inductive argument, we prove that the Gomory-Chvátal closure of any polytope is a rational polytope.

Theorem 4.24 *The Gomory-Chvátal closure P' of a non-rational polytope P is a rational polytope.*

Proof. The proof is by induction on the dimension $d \leq n$ of $P \subseteq \mathbb{R}^n$. Let $n \geq 1$ be arbitrary. The base case, $d = 0$, is trivially true. Therefore, assume that $d \geq 1$. By Corollary 4.18, we know that there exists a finite set $S_1 \subseteq \mathbb{Z}^n$ such that

$$C_{S_1}(P) \subseteq P \ .$$

Let $\{F_i\}_{i \in I}$ denote the set of facets of P and assume that $F^i = P \cap (a^i x = a^i_P)$. By the induction assumption for $d-1$, we know that F_i' is a rational polytope for every $i \in I$. That is, there exists a finite set $S_i \subseteq \mathbb{Z}^n$ such that

$$C_{S_i}(F_i) = F_i' \ .$$

According to Lemma 4.21, we can find for every Gomory-Chvátal cut for F_i that is facet-defining for F_i' a Gomory-Chvátal cut for P that has the same impact on the maximal rational affine subspace of $(a^i x = a^i_P)$. Furthermore, by Corollary 4.17, F_i' is contained in this rational affine subspace. Hence, for every $i \in I$, there exists a finite set $\bar{S}_i \subseteq \mathbb{Z}^n$ such that

$$C_{\bar{S}_i}(P) \cap F_i = F_i' \ .$$

Because $\mathrm{rbd}(P) = \cup_{i \in I} F_i$, the set $S = S_1 \cup \big(\cup_{i \in I} \bar{S}_i \big)$ satisfies

$$\begin{aligned} C_S(P) &\subseteq P \ , \\ C_S(P) \cap \mathrm{rbd}(P) &\subseteq P' \ . \end{aligned}$$

By Lemma 4.23, P' is a rational polytope. $\square$

Chapter 5

A Refined Gomory-Chvátal Closure for Polytopes in the Unit Cube

5.1 Introduction

Numerous cutting planes for integer and mixed-integer programs have been introduced and studied in the literature (see, e.g., Cornuéjols and Li 2000 and Cornuéjols 2008). One possible way to classify general cutting planes – that is, cutting planes that are derived solely from the linear description of a polyhedron without knowing any specific structure of the underlying problem – is according to the types of polyhedra that they can be applied to. For example, there exist families of cutting planes that are derived for general (mixed) integer programming problems in $\mathbb{R}^n$ (such as Gomory mixed integer cuts or disjunctive cuts), while others require that the integer hull of the given polytope is contained in the unit cube (e.g., lift-and-project cuts or knapsack cuts). The more general the derivation of a family of cutting planes, in the sense that fewer assumptions are made about the linear programming relaxation or its integer hull, the wider the class of problems that they can be applied to. On the other hand, when applying a cutting plane method that was designed for a broad class of problems to a specific subclass of instances, not all the structural properties that distinguish this subclass from the general case are exploited. Consider for example the Gomory-Chvátal procedure, a method designed for arbitrary integer programming problems. Suppose that we apply the procedure to a polytope $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ for which all variables have implicit lower and upper bounds; in particular, every integer point in P is contained in some cube $[L, U]^n$, where $L \leq U$. The standard Gomory-Chvátal procedure takes every valid inequality $ax \leq a_0$ for P such that a is an integral vector with relatively prime components and rounds down the right-hand side a_0. Geometrically, the Gomory-Chvátal cutting plane is obtained by shifting the boundary of the half-space $(ax \leq a_0)$ towards the polytope until it intersects an integral point. The procedure guarantees that every integer point in $\mathbb{R}^n$ that satisfies the original inequality also satisfies the strengthened one. In fact, for the purpose of the given problem it would suffice to require only that every integer point in $[L, U]^n \cap (ax \leq a_0)$ satisfies the modified inequality. This alone guarantees that no integer point in P is cut off. From a geometric point of view, we can shift the boundary of the half-space $(ax \leq a_0)$ until it contains an integer point of the cube $[L, U]^n$ and, thus, ignore any integer points outside this cube. As a result, we obtain a valid inequality for the integer hull P_I that potentially dominates the corresponding Gomory-Chvátal cut.

In this chapter, we formalize this idea for the important class of 0/1 integer programming problems. Many combinatorial optimization problems can be modeled as integer programs with decision variables, that is, variables that can take the value 0 or 1, depending on the occurrence of a particular event. The standard relaxations of these problems are optimization problems over polytopes that are subsets of the unit cube $[0, 1]^n$. We introduce a refined family of Gomory-Chvátal cutting planes for this special case. More precisely, we define the *M-cut* associated with a valid inequality $ax \leq a_0$ for a polytope $P \subseteq [0, 1]^n$ to be the inequality $ax \leq a_0'$, which results from decreasing the right-hand side a_0 until the hyperplane $(ax = a_0')$ contains a 0/1 point. Hence, the M-cut associated with a valid inequality generally dominates the corresponding Gomory-Chvátal cut. We call the implied cutting plane method the *M-procedure*. It represents a natural strengthening of the Gomory-Chvátal procedure in the sense that the right-hand sides of Gomory-Chvátal cuts are further decreased if the boundaries of the associated half-spaces do not intersect any 0/1 points. Hence, when shifting the boundaries of half-spaces containing P, we neglect the integer points outside the unit cube. Similar to the Gomory-Chvátal closure for a polytope P, we define the *M-closure* of P as the intersection of all half-spaces implied by the M-cuts of P. We denote this closure by $M(P)$. As it is the case for the elementary closure, the set of M-cuts of a given polytope is infinite. For that reason, it is not obvious whether the M-closure of a polytope is again a polytope. However, we show that for any polytope $P \subseteq [0,1]^n$ a finite number of M-cuts suffice to describe $M(P)$. In addition, we study various other structural properties of this closure. For example, we discover an interesting characteristic regarding facet-defining inequalities for the case that $M(P)$ is full-dimensional. While it follows directly from the definition of the M-closure that every undominated inequality for $M(P)$ is tight at a 0/1 point, we show that any facet-defining inequality $ax \leq a_0$ for $M(P)$ corresponds to a hyperplane ($ax = a_0$) that is *spanned* by 0/1 points. In other words, the M-cut is tight at n affinely independent points in $\{0,1\}^n$. This property can be seen as analogous to the fact that every undominated Gomory-Chvátal cut is associated with a hyperplane that is spanned by n affinely independent integer points. Furthermore, the M-closure shares an important property with P': even though the right-hand sides of M-cuts are derived in a seemingly less structured way, compared to the consistent rounding in the Gomory-Chvátal procedure, for any face F of P, it holds that $M(F) = M(P) \cap F$.

Cornuéjols and Li (2000) gave a detailed overview and comparison of the closures associated with certain families of cutting planes for 0/1 integer programs. They established all inclusion relationships between these closures. Surprisingly, various of the seemingly different closures derived from different types of cutting planes turned out to be equivalent. For example, the intersection of all half-spaces associated with lift-and-project cuts of a polytope yields the same set as the intersection of the half-spaces implied by simple disjunctive cuts. We study the relationship between the M-closure of a polytope and these well-known other closures, complementing the picture drawn by Cornuéjols and Li.

Moreover, we investigate the sequence of successively tighter approximations of the integer hull of a rational polytope arising from a repeated application of all M-cuts to the polytope. Clearly, as the M-closure of a polytope P in the unit cube $[0,1]^n$ is contained in P', any such sequence will result in P_I after polynomially many steps (see Theorem 2.19). We show that if the integer hull of a polytope is empty, then the M-procedure requires, in the worst case, as many iterations as the Gomory-Chvátal procedure to generate an empty polytope.

Finally, we study complexity questions related to the M-closure. We show that, in fixed dimension, the M-closure can be computed in polynomial time. In general, however, the membership problem is NP-hard.

5.2 Outline

In Section 5.3, we begin by formally introducing M-cuts and the M-closure of a polytope. In Section 5.3.1, we investigate structural properties of the M-closure. A detailed comparison of the M-closure with other known elementary closures from the literature follows in Section 5.4. In Section 5.5, we study the cutting plane procedure associated with M-cuts. More precisely, we analyze how quickly an iterative application of all M-cuts will generate the integer hull of a given polytope. Finally, Section 5.6 is concerned with the complexity of the M-closure.

5.3 The M-Closure of a Polytope

Throughout this chapter, we consider polytopes $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ whose integer points are contained in the n-dimensional unit cube $[0,1]^n$. As in this case the inequalities $0 \leq x_i \leq 1$, for $i = 1, \ldots, n$, are known to be valid for the integer hull P_I, we can assume w.l.o.g. that

$$P \subseteq [0,1]^n \quad . \tag{5.1}$$

For any vector $a \in \mathbb{Z}^n$, we define a_P to be the smallest number such that $ax \leq a_P$ is a valid inequality for P, that is,

$$a_P := \max \{ax \mid x \in P\} \quad .$$

Consider an arbitrary inequality $ax \leq a_0$ such that the polytope P is contained in the half-space $H = (ax \leq a_0)$. If $a \in \mathbb{Z}^n$ and $\gcd(a_1, \ldots, a_n) = 1$, then $(ax = a_0) \cap \mathbb{Z}^n \neq \emptyset$ if and only if $a_0 \in \mathbb{Z}$. In particular, $(ax \leq \lfloor a_0 \rfloor)$ is the integer hull of the half-space H. Clearly, as P_I is contained in H_I, the inequality $ax \leq \lfloor a_0 \rfloor$ is a cutting plane for P. It is the classic Gomory-Chvátal cut associated with the valid inquality $ax \leq a_0$. Now suppose that $(ax = \lfloor a_0 \rfloor) \cap \{0,1\}^n = \emptyset$. Then assumption (5.1) implies that $P_I \subseteq (ax \leq \lfloor a_0 \rfloor - 1)$. Hence, we can further decrease the right-hand side $\lfloor a_0 \rfloor$ and, yet, guarantee that the resulting inequality does not separate any integer point from P. Let us define the *knapsack* *value* for any pair $(a, a_0) \in \mathbb{Q}^n \times \mathbb{R}$ as

$$KV(a, a_0) := \max \left\{ ax \mid x \in \{0,1\}^n, ax \le a_0 \right\} \ ,$$

where we set $KV(a, a_0) := -\infty$ if $(ax \le a_0) \cap \{0,1\}^n = \emptyset$. Then it holds that

$$P_I \subseteq \big(ax \le KV(a, a_0) \big) \ ,$$

and the inequality $ax \le KV(a, a_0)$ is a cutting plane for P. Geometrically speaking, this cut is obtained by shifting the hyperplane $(ax = a_0)$ towards the polytope until a 0/1 point is contained in it (see Figure 5-1). The set of 0/1 points satisfying the inequality $ax \le a_0$ is the same as the set of 0/1 points satisfying the cut $ax \le KV(a, a_0)$.

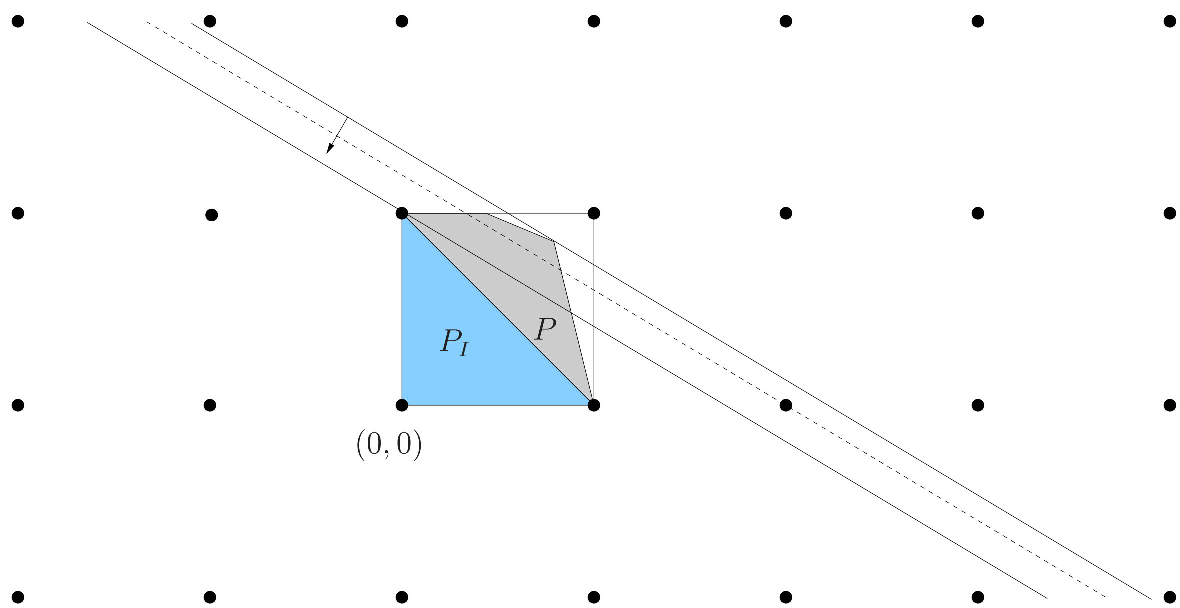

Figure 5-1: The inequality $3x_1 + 5x_2 \le 7 - \varepsilon$ is valid for $P \subseteq [0,1]^n$, for some $\varepsilon > 0$. The hyperplane $3x_1 + 5x_2 = 6$ does not contain a 0/1 point, but the hyperplane $3x_1 + 5x_2 = 5$ does. The inequality $3x_1 + 5x_2 \le 5$ is valid for P_I and dominates the corresponding Gomory-Chvátal cut $3x_1 + 5x_2 \le 6$.

For any $a \in \mathbb{Q}^n$, the inequality $ax \leq KV(a, a_P)$ is called the *M-cut associated with the valid inequality* $ax \leq a_P$. Note that there is an infinite set of M-cuts for a polytope that define the same half-space. The simultaneous application of all M-cuts to a polytope gives rise to a new elementary closure: we define the M-closure $M(P)$ of a polytope $P \subseteq [0, 1]^n$ as

$$M(P) := \bigcap_{a \in \mathbb{Z}^n} \Big(ax \leq KV(a, a_P) \Big) \ . \tag{5.2}$$

Note that in the definition of the M-closure, we could restrict the intersection to M-cuts $ax \leq KV(a, a_P)$ for which $a \in \mathbb{Z}^n$, since every rational half-space has a representation with an integral normal vector. In analogy to Section 2.7.2, we define the *M-procedure* as the iterative application of the M-closure operation to a polytope. More precisely, we define $M^{(0)}(P) := P$ and $M^{(k+1)}(P) := M(M^{(k)}(P))$, for every integer $k \geq 0$. As a result, we obtain a sequence

$$P \supseteq M(P) \supseteq M^{(2)}(P) \supseteq \ldots \supseteq P_I$$

of successively tighter approximations of P_I. We call the smallest number $t \in \mathbb{N}$ such that $M^{(t)}(P) = P_I$ the *M-rank* of P.

5.3.1 Structural Properties

As elementary closures of polytopes are defined with respect to families of cutting planes that have a clear methodical derivation, they are generally characterized by a list of natural structural properties. Of foremost importance, when regarding the main purpose of cutting planes, is the property that an iterative application of a closure operation to a polytope generates its integer hull after a finite number of steps. The property of monotonicity is also intuitive: if the closure operation is applied to a pair of polytopes, such that one is contained in the other, the same inclusion relation is satisfied by their closures. In particular, given two relaxations P_1 and P_2 of a 0/1 polytope such $P_1 \subseteq P_2$, no more iterations will be needed to generate the integer hull from P_1 compared to the number of iterations that are required when starting with the relaxation P_2.

In the following lemma, we prove some basic properties for the M-closure of a general polytope (that is, these properties also hold for polytopes that cannot be described by rational data).

Lemma 5.1 *Let P and Q be polytopes in $[0,1]^n$. Then the following properties hold: The M-closure*

(1) approximates the integer hull: $P_I \subseteq M(P) \subseteq P$.

(2) preserves inclusion: If $P \subseteq Q$, then $M(P) \subseteq M(Q)$.

(3) rounds coordinates: If $x_i \leq \varepsilon$ ($x_i \geq \varepsilon$) is valid for P for some $0 < \varepsilon < 1$, then $x_i \leq 0$ ($x_i \geq 1$) is valid for $M(P)$.

(4) commutes with coordinate flips: Let $\tau_i : [0,1]^n \to [0,1]^n$ with $x_i \mapsto (1 - x_i)$ be a coordinate flip. Then $\tau_i(M(P)) = M(\tau_i(P))$.

Proof. (1) $P_I \subseteq M(P)$ follows from the observation that the set of 0/1 points satisfying an M-cut is the same as the set of 0/1 points satisfying the valid inequality of P, from which the M-cut was derived. The second inclusion follows for rational polytopes from the fact that every rational valid inequality for P is dominated by its corresponding M-cut. If P is a non-rational polytope, then Corollary 4.18 implies that $M(P) \subseteq P' \subseteq P$.

(2) As every valid inequality for Q is also valid for P, every M-cut for Q is also an M-cut for P.

(3) The inequality $x_i \leq 0$ ($x_i \geq 1$) is the M-cut associated with the valid inequality $x_i \leq \varepsilon$ ($x_i \geq \varepsilon$).

(4) This property follows from the symmetry of the unit cube.

□

Next, we focus our attention to a less obvious property that we already studied with respect to the Gomory-Chvátal closure in Section 4.4.3. There, we showed that for any face F of a polytope P, $F' = P' \cap F$. While this property had been well-known for rational polytopes, we extended the result to general polytopes. In particular, we showed that every Gomory-Chvátal cutting plane for F can be rotated to become a Gomory-Chvátal cutting plane for P that has the same impact on the maximal rational affine subspace of any hyperplane defining F. In the next lemma, we prove that this *homogeneity* property is also satisfied for the M-closure of a rational polytope.

Lemma 5.2 *Let $P \subseteq [0,1]^n$ be a rational polytope and let F be a face of P. Then*

$$M(F) = M(P) \cap F \ .$$

Proof. Suppose that $P = \{x \in \mathbb{R}^n \mid Ax \leq b\} \neq \emptyset$, where $A \in \mathbb{Z}^{m\times n}$ and $b \in \mathbb{Z}^m$. Furthermore, assume that $F = P \cap (ax = a_P) \neq \emptyset$, where $a \in \mathbb{Z}^n$, $\gcd(a) = 1$, and $P \subseteq (ax \leq a_P)$. The inclusion $M(F) \subseteq M(P) \cap F$ is obviously true, since any inequality that is valid for P is also valid for F and since $M(F) \subseteq F$.

In the remainder of the proof, we show $M(P) \cap F \subseteq M(F)$. If $(ax = a_P) \cap \{0,1\}^n = \emptyset$, then $K(a, a_P) < a_P$ and the M-cut associated with $ax \leq a_P$ strictly dominates $ax \leq a_P$. Consequently, $M(P) \cap F = \emptyset \subseteq M(F)$. Therefore, let us assume in the following that

$$(ax = a_P) \cap \{0,1\}^n \neq \emptyset \ . \tag{5.3}$$

Consider an arbitrary M-cut $cx \leq KV(c, c_F)$ for F that is associated with the valid inequality $cx \leq c_F$ for F. From Farkas' Lemma 2.5, we know that there exist a vector $\lambda \in \mathbb{Q}^m_+$ and a number $\mu \in \mathbb{Q}$ such that $c = \lambda A + \mu a$ and $c_F \geq \lambda b + \mu a_P$. If $\mu \geq 0$, then $cx \leq c_F$ is also valid for P, implying that $cx \leq KV(c, c_F)$ is an M-cut for P. We therefore assume in the following that $\mu < 0$. Consider the set

$$S = \left\{x \in \{0,1\}^n \,\middle|\, ax = a_P,\ \lambda A x \leq \lambda b\right\} \ .$$

We will distinguish two cases. First, assume $S = \emptyset$. Let

$$\begin{aligned}
M_1 &:= \min\left\{\lambda A x \,\middle|\, x \in [0,1]^n,\ ax = a_P\right\} \ , \\
M_2 &:= \min\left\{\lambda A x \,\middle|\, x \in \{0,1\}^n,\ ax \geq a_P + 1\right\} \ , \\
M_3 &:= \max\left\{\lambda A x \,\middle|\, x \in \{0,1\}^n,\ ax \leq a_P - 1\right\} \ .
\end{aligned}$$

Note that M_1 has a finite value, since $F \neq \emptyset$. Furthermore, $M_2 \in (-\infty, \infty]$

and $M_3 \in [-\infty, \infty)$. Therefore, we can find a finite number ν that satisfies

$$\nu > \max\{\lambda b - M_2, M_3 - M_1, 0\} \ . \tag{5.4}$$

By Lemma 2.5, the inequality $\bar{c}x \le \bar{c}_0$ with $\bar{c} := \lambda A + \nu a$ and $\bar{c}_0 := \lambda b + \nu a_P$ is valid for P. If $M_2 = \infty$, then $\{0,1\}^n \cap (ax \ge a_P + 1) = \emptyset$. Otherwise, (5.4) implies for every $x \in \{0,1\}^n \cap (ax \ge a_P + 1)$ that

$$\bar{c}x = \lambda Ax + \nu ax \ge M_2 + \nu + \nu a_P > \lambda b + \nu a_P = \bar{c}_0 \ .$$

That is, every such point violates the inequality $\bar{c}x \le \bar{c}_0$. Consequently, and together with the assumption $S = \emptyset$, we obtain

$$\{0,1\}^n \cap (ax \ge a_P) \cap (\bar{c}x \le \bar{c}_0) = \emptyset \ . \tag{5.5}$$

Hence, the right-hand side of the M-cut for P associated with $\bar{c}x \le \bar{c}_0$ satisfies

$$\begin{aligned} KV(\bar{c}, \bar{c}_0) &= \max\left\{\bar{c}x \,\middle|\, x \in \{0,1\}^n,\, \bar{c}x \le \bar{c}_0\right\} \\ &= \max\left\{(\lambda A + \nu a)\, x \,\middle|\, x \in \{0,1\}^n,\ ax \le a_P - 1,\ (\lambda A + \nu a)\, x \le \lambda b + \nu a_P\right\} \ . \end{aligned}$$

Note, that

$$\{0,1\}^n \cap (\bar{c}x \le \bar{c}_0) \ne \emptyset \ ,$$

since, otherwise, the fact that $\bar{c}x \le \bar{c}_0$ is valid for P would imply $P = \emptyset$. Hence, we have

$$\{0,1\}^n \cap (\bar{c}x \le \bar{c}_0) = \{0,1\}^n \cap (ax \le a_P - 1) \cap \big((\lambda A + \nu a)\, x \le \lambda b + \nu a_P\big) \ne \emptyset \ ,$$

which implies that M_3 is finite. We obtain

$$\begin{aligned} KV(\bar{c}, \bar{c}_0) &\le \max\left\{(\lambda A + \nu a)\, x \,\middle|\, x \in \{0,1\}^n,\ ax \le a_P - 1\right\} \\ &\le \max\left\{\lambda Ax \mid x \in \{0,1\}^n,\ ax \le a_P - 1\right\} + \nu(a_P - 1) \\ &= M_3 + \nu(a_P - 1) \ . \end{aligned}$$

As a result, $(\lambda A + \nu a)\, x \le M_3 + \nu(a_P - 1)$ is valid for $M(P)$. For an arbitrary point $x \in [0,1]^n \cap (ax = a_P)$, (5.4) implies that

$$\bar{c}x = (\lambda A + \nu a)\, x = \lambda A x + \nu a_P \geq (M_1 + \nu) + \nu(a_P - 1) > M_3 + \nu(a_P - 1) > KV(\bar{c}, \bar{c}_0) \ ,$$

that is, any point in F violates the M-cut $\bar{c}x \leq KV(\bar{c}, \bar{c}_0)$ for P. This implies that

$$M(P) \cap F = \emptyset \subseteq M(F) \ .$$

Now consider the second case: $S \neq \emptyset$. We define two more constants

$$\begin{aligned} M_4 &:= \min\left\{\lambda A x \,\middle|\, x \in \{0,1\}^n\right\} \ , \\ M_5 &:= \max\left\{\lambda A x \,\middle|\, x \in \{0,1\}^n,\ ax = a_P,\ \lambda A x \leq \lambda b\right\} \ . \end{aligned}$$

Observe, that both constants are finite because of the assumption $S \neq \emptyset$. Now we can choose ν to be finite and such that

$$\nu > \max\{\lambda b - M_4, M_3 - M_5, 0\} \ . \tag{5.6}$$

Again by Lemma 2.5, the inequality $\bar{c}x \leq \bar{c}_0$ with $\bar{c} := \lambda A + \nu a$ and $\bar{c}_0 := \lambda b + \nu a_P$ is valid for P. With the definition of M_4 and condition (5.6), we get for any point $x \in \{0,1\}^n \cap (ax \geq a_P + 1)$,

$$\bar{c}x = \lambda A x + \nu a x \geq M_4 + \nu + \nu a_P > \lambda b + \nu a_P = \bar{c}_0 \ .$$

It follows that

$$\{0,1\}^n \cap (ax \geq a_P + 1) \cap (\bar{c}x \leq \bar{c}_0) = \emptyset \ . \tag{5.7}$$

Observe, that

$$\{0,1\}^n \cap (ax = a_P) \cap (\bar{c}x \leq \bar{c}_0) = S \neq \emptyset \ . \tag{5.8}$$

If

$$\{0,1\}^n \cap (ax \leq a_P - 1) \cap (\bar{c}x \leq \bar{c}_0) \neq \emptyset \ ,$$

then M_3 is finite and we get with (5.6),

$$
\begin{aligned}
& \max\left\{\bar{c}x \,\middle|\, x \in \{0,1\}^n,\ ax \le a_P - 1,\ \bar{c}\,x \le \bar{c}_0\right\} \\
= \ & \max\left\{(\lambda A + \nu a)\,x \,\middle|\, x \in \{0,1\}^n,\ ax \le a_P - 1,\ (\lambda A + \nu a)\,x \le \lambda b + \nu a_P\right\} \\
\le \ & \max\left\{(\lambda A + \nu a)\,x \,\middle|\, x \in \{0,1\}^n,\ ax \le a_P - 1\right\} \\
\le \ & \max\left\{\lambda A x \,\middle|\, x \in \{0,1\}^n,\ ax \le a_P - 1\right\} + \nu(a_P - 1) \\
= \ & (M_3 - \nu) + \nu a_P \\
< \ & M_5 + \nu a_P \\
= \ & \max\left\{\lambda A x \,\middle|\, x \in \{0,1\}^n,\ ax = a_P,\ \lambda A x \le \lambda b\right\} + \nu a_P \\
= \ & \max\left\{(\lambda A + \nu a)\,x \,\middle|\, x \in \{0,1\}^n,\ ax = a_P,\ (\lambda A + \nu a)\,x \le \lambda b + \nu a_P\right\} \\
= \ & \max\left\{\bar{c}x \,\middle|\, x \in \{0,1\}^n,\ ax = a_P,\ \bar{c}x \le \bar{c}_0\right\} \ .
\end{aligned}
$$

Because of (5.7), (5.8), and the last observation, the maximum in the computation of $KV(\bar{c}, \bar{c}_0)$ is achieved by a 0/1 point in $(ax = a_P)$. In particular,

$$
\begin{aligned}
KV(\bar{c}, \bar{c}_0) &= \max\{\bar{c}x \mid x \in \{0,1\}^n,\ \bar{c}x \le \bar{c}_0\} \\
&= \max\{(\lambda A + \nu a)\,x \mid x \in \{0,1\}^n,\ (ax = a_P),\ (\lambda A + \nu a)\,x \le \lambda b + \nu a_P\} \\
&= \max\{\lambda A x \mid x \in \{0,1\}^n,\ (ax = a_P),\ \lambda A x \le \lambda b\} + \nu a_P \\
&= M_5 + \nu a_P \ .
\end{aligned}
$$

On the other hand, the right-hand side of the M-cut $cx \le K(c, c_F)$ for F satisfies

$$
\begin{aligned}
KV(c, c_F) &= \max\{cx \,|\, x \in \{0,1\}^n,\ cx \le c_F\} \\
&\ge \max\left\{(\lambda A + \mu a)\,x \,\middle|\, x \in \{0,1\}^n,\ (\lambda A + \mu a)\,x \le \lambda b + \mu a_P\right\} \\
&\ge \max\left\{(\lambda A + \mu a)\,x \,\middle|\, x \in \{0,1\}^n,\ ax = a_P,\ (\lambda A + \mu a)\,x \le \lambda b + \mu a_P\right\} \\
&= \max\left\{\lambda A x \,\middle|\, x \in \{0,1\}^n,\ ax = a_P,\ \lambda A x \le \lambda b\right\} + \mu a_P \\
&= M_5 + \mu a_P \ .
\end{aligned}
$$

As a result, we obtain

$$\begin{aligned}\left(\bar{c}x \le KV(\bar{c},\bar{c}_0)\right) \cap (ax = a_P) &= \left((\lambda A + \nu a)\, x \le M_5 + \nu a_P\right) \cap (ax = a_P)\\ &= (\lambda A x \le M_5) \cap (ax = a_P)\\ &= \left((\lambda A + \mu a)\, x \le M_5 + \mu a_P\right) \cap (ax = a_P)\\ &\subseteq \left(cx \le KV(c, c_F)\right) \cap (ax = a_P)\ ,\end{aligned}$$

and the M-cut $\bar{c}x \le KV(\bar{c},\bar{c}_0)$ for P dominates the M-cut $cx \le KV(c, c_F)$ for F in F. This completes the proof. $\square$

5.3.2 Polyhedrality

Similar to the Gomory-Chvátal closure, the M-closure (5.2) of a polytope P is defined as the intersection of an infinite number of half-spaces. In the case of the Gomory-Chvátal closure, it is well-known that, instead of looking at every valid inequality for P, it suffices to consider only those normal vectors which are associated with the Hilbert bases of the basic feasible cones of P and their number is finite. Equivalently, the elementary closure P' can be derived from a totally dual integral system $Ax \le b$ describing P, such that A is an integral matrix. In that case, P' is obtained by rounding down the right-hand side vector, that is, $P' = \{x \in \mathbb{R}^n \mid Ax \le \lfloor b \rfloor\}$. Naturally arises the question as to whether the M-closure of a polytope can also be described by a finite system of inequalities, that is, whether $M(P)$ is a polytope itself. Here, we want to examine this question for rational polytopes.

The way in which the M-procedure decreases the right-hand sides of valid inequalities is much less structured than the rounding operation for Gomory-Chvátal cuts. While for some inequalities the right-hand side is simply rounded down, as is the case for the Gomory-Chvátal procedure, there are M-cuts that strictly dominate the corresponding Gomory-Chvátal cut. Hence, the reasoning behind the proof that the Gomory-Chvátal closure of a rational polytope is also a polytope cannot be applied for the M-closure. Yet, we show that $M(P)$ is indeed a polytope for every rational P. The general idea of our proof is to represent the M-cuts of P as feasible points of a finite collection of rational polyhedra. Since every M-cut induces a partition of the set of 0/1 points into three sets – the set of 0/1 points that satisfy the cut with strict inequality, the set of $0/1$ points at which the cut is tight, and the set of $0/1$ points that violated the M-cut – it is possible to partition the infinite set of M-cuts into a finite number of subsets that are associated with these partitions. For each of these sets, we will construct a single polyhedron such that every feasible point of that polyhedron corresponds to an M-cut in the set, and vice versa. We then use the decomposition theorem for polyhedra to show that every M-cut is dominated by a finite set of M-cuts that correspond to the basic feasible solutions and extreme rays of the polyhedron that the M-cut is associated with.

First, let us introduce some notation. Let (L, E, G) with $L \dot{\cup} E \dot{\cup} G = \{0,1\}^n$ be a partition of the set of $0/1$ points in $\mathbb{R}^n$. We will denote by $M_{(L,E,G)}(P)$ the set of all pairs $(a, a_0) \in \mathbb{Q}^{n+1}$ that correspond to M-cuts $ax \leq a_0$ for P for which

$$\begin{array}{rcll} ax & < & a_0 & \text{for all } x \in L \ , \\ ax & = & a_0 & \text{for all } x \in E \ , \\ ax & > & a_0 & \text{for all } x \in G \ . \end{array}$$

Thus, every M-cut associated with the set $M_{(L,E,G)}(P)$ partitions the set of $0/1$ points into the three sets L, E, and G. Every point in L is strictly contained in the half-space $(ax \leq a_0)$, every point in E is on its boundary $(ax = a_0)$, and every point in G violates the inequality. As every rational half-space has an infinite number of representations, $M_{(L,E,G)}(P)$ contains an infinite number of representatives of the same half-space.

Since every M-cut of P is associated with a certain partition of the $0/1$ points in $\mathbb{R}^n$, the union of the sets $M_{(L,E,G)}(P)$ over all possible partitions (L, E, G) of $\{0,1\}^n$ completely describes $M(P)$. In other words, $M(P)$ is the intersection of all inequalities associated with the finite number of sets $M_{(L,E,G)}(P)$. We will first show that for every partition (L, E, G), the set $M_{(L,E,G)}(P)$ can be represented as the feasible set of a rational polyhedron.

Lemma 5.3 *Let $P \subseteq [0,1]^n$ be a non-empty rational polytope and let $\mathcal{V}(P)$ denote its set of extreme points. Furthermore, let (L, E, G) be an arbitrary partition of $\{0,1\}^n$*

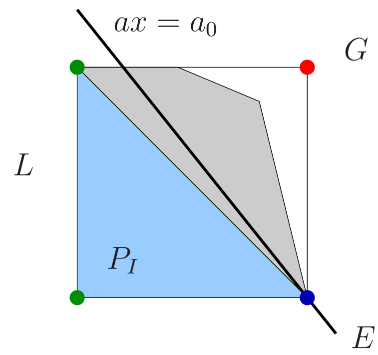

Figure 5-2: Classification of M-cuts according to the implied partition of 0/1 points into three sets L, E, and G.

with $L, E, G \neq \emptyset$, and let $0 < \delta \in \mathbb{Q}$ be a small constant. If (a, a_0) is a pair in $M_{(L,E,G)}(P)$, then there exists a constant $\lambda > 0$ such that $(\lambda a, \lambda a_0)$ is a feasible point of the polyhedron $Q_{(L,E,G)}$ that is defined by the following constraints in variables $(\alpha, \alpha_0) \in \mathbb{R}^{n+1}$:

$$
\begin{aligned}
\alpha x &= \alpha_0 && \text{for all } x \in E && (Q_{(L,E,G)}(P))\\
\alpha x &\geq \alpha_0 + \delta && \text{for all } x \in G\\
\alpha x &\leq \alpha_0 - \delta && \text{for all } x \in L\\
\alpha z &\leq \alpha x - \delta && \text{for all } x \in G \text{ and for all } z \in \mathcal{V}(P) \ .
\end{aligned}
$$

Conversely, every rational point (α, α_0) in $Q_{(L,E,G)}(P)$ is a pair in $M_{(L,E,G)}(P)$.

Proof. For the first part of the lemma, assume that (a, a_0) is a pair in $M_{(L,E,G)}(P)$. That is, $ax \leq a_0$ is an M-cut for P that is tight at all 0/1 points in E, violated by all points in G and satisfied with strict inequality by the points in L. Then there exist positive constants Δ_1, Δ_2, and Δ_3 such that

$$
\begin{aligned}
K_L &:= \max\{ax \mid x \in L\} \leq a_0 - \Delta_1 < a_0 + \Delta_2 \leq \min\{ax \mid x \in G\} =: K_G \ ,\\
K_z &:= \max\{az \mid z \in \mathcal{V}(P)\} \leq \min\{ax \mid x \in G\} - \Delta_3 = K_G - \Delta_3 \ .
\end{aligned}
$$

Note that the constants K_L, K_G, and K_z are finite (see Figure 5-3 for an illustration). For every $\lambda > 0$ satisfying $\lambda \Delta_i \geq \max\{\alpha_0, \delta\}$ for $i = 1, 2, 3$, it holds that

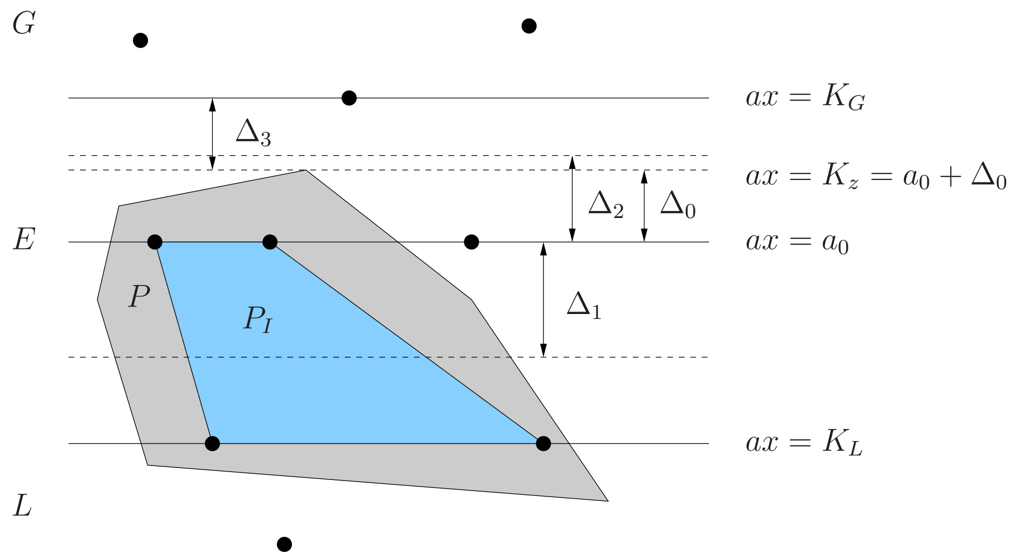

Figure 5-3: The inequality $ax \leq a_0$ is an M-cut for P that is associated with the valid inequality $ax \leq a_0 + \Delta_0$ of P, where $\Delta_0 \geq 0$. The set of points in $(ax = a_0)$ is precisely E, all points in G lie strictly above $(ax = a_0 + \Delta_0)$, all points in L lie strictly below $(ax = a_0)$. (Note: Only 0/1 points are illustrated and no other integer points.)

$$\begin{aligned} a_0 \leq \quad \lambda\Delta_1 \quad &\leq \quad \lambda(a_0 - K_L) \ , \\ a_0 \leq \quad \lambda\Delta_2 \quad &\leq \quad \lambda(K_G - a_0) \ , \\ a_0 \leq \quad \lambda\Delta_3 \quad &\leq \quad \lambda(K_G - K_z) \ , \end{aligned}$$

and therefore

$$\begin{aligned} \delta \quad &\leq \quad \lambda(a_0 - K_L) \quad = \quad \lambda a_0 - \max\{\lambda ax \mid x \in L\} \ , \\ \delta \quad &\leq \quad \lambda(K_G - a_0) \quad = \quad \min\{\lambda ax \mid x \in G\} - \lambda a_0 \ , \\ \delta \quad &\leq \quad \lambda(K_G - K_z) \quad = \quad \min\{\lambda ax \mid x \in G\} - \max\{\lambda az \mid z \in \mathcal{V}(P)\} \ . \end{aligned}$$

Consequently, $(\lambda a, \lambda a_0)$ is a point in $Q_{(L,E,G)}(P)$.

Now consider the second part of the lemma. If (α, α_0) is a rational point in $Q_{(L,E,G)}(P)$, then for $a := \alpha$ and $a_0 := \alpha_0$, $ax \leq a_0$ is the M-cut associated with the valid inequality $ax \leq a_0 + \Delta_0$, where $\Delta_0 := \max\{ax \mid x \in P\} - a_0$. This is, since for every 0/1 point $x \in G$, we have

$$ax \geq \max\{az \mid z \in \mathcal{V}(P)\} + \delta = a_0 + \Delta_0 + \delta > a_0 + \Delta_0 \ ,$$

and thus,

$$\begin{aligned} a_0 &= \max\left\{ax \,\middle|\, x \in \{0,1\}^n,\ x \in (L \cup E),\ ax \leq a_0 + \Delta_0\right\} \\ &= \max\left\{ax \,\middle|\, x \in \{0,1\}^n,\ ax \leq a_0 + \Delta_0\right\} \\ &= KV(a, a_0 + \Delta_0) \ . \end{aligned}$$

□

Note that the feasible set of $Q_{(L,E,G)}(P)$ is either empty or unbounded, since any feasible solution can be scaled by an arbitrary positive constant greater than 1 to obtain another feasible solution. More precisely, $Q_{(L,E,G)}(P)$ is non-empty if and only if there exists a hyperplane that contains every point in E and that strictly separates conv(G) from conv(L).

In order to show that $M(P)$ can be described by a finite set of inequalities, we want to make use of the fact that the feasible set of a polyhedron can be represented as the sum of the convex hull of its vertices plus the cone generated by its extreme rays. Lemma 5.3 implies that any rational point in $Q_{(L,E,G)}(P)$, and in particular any vertex of $Q_{(L,E,G)}(P)$, defines a valid inequality for $M(P)$. In the following lemma, we show that also every extreme ray of $Q_{(L,E,G)}(P)$ is associated with a valid inequality for $M(P)$.

Lemma 5.4 *If $Q_{(L,E,G)}(P)$ is feasible and (r, r_0) an extreme ray of $Q_{(L,E,G)}(P)$, then $rx \leq r_0$ defines a valid inequality for $M(P)$.*

Proof. Let (r, r_0) be an extreme ray of $Q_{(L,E,G)}(P)$ and suppose that there exists some $\bar{x} \in M(P)$ with $r\,\bar{x} > r_0$. Define $\gamma := r\bar{x} - r_0 > 0$. Now consider an arbitrary rational point $(\alpha, \alpha_0) \in Q_{(L,E,G)}(P)$. Then $(\alpha, \alpha_0) + \lambda(r, r_0)$ is in $Q_{(L,E,G)}(P)$, for any $\lambda \geq 0$. By Lemma 5.3, it follows that $(\alpha + \lambda r)\,x \leq \alpha_0 + \lambda r_0$ is a valid inequality for $M(P)$ for every rational $\lambda \geq 0$. In particular, $(\alpha + \lambda r)\,\bar{x} \leq \alpha_0 + \lambda r_0$. We obtain $\lambda\gamma \leq \alpha_0 - \alpha\bar{x}$ for any $\lambda \geq 0$. However, since the right-hand side of the last inequality is finite, there exists some rational λ such that $\lambda\gamma > a_0 - a\bar{x}$. This is a contradiction to the fact that $\bar{x} \in M(P)$ and, therefore, $r\,\bar{x} \leq r_0$ must hold. $\square$

With this, we are prepared to show that the M-closure of any rational polytope can be described by a finite set of inequalities.

Theorem 5.5 *If $P \subseteq [0,1]^n$ is a rational polytope, then $M(P)$ is a rational polytope.*

Proof. If $P = \emptyset$, there is nothing to show. Therefore, assume that $P \neq \emptyset$. First, observe that $P \neq \emptyset$ implies that there is no M-cut for P that belongs to a set $M_{(L,E,G)}(P)$ with $E = \emptyset$. Let $M_{(L=\emptyset)}(P)$ denote the union of all sets $M_{(L,E,G)}(P)$ for which $L = \emptyset$. For any cut $ax \leq a_0$ in $M_{(L=\emptyset)}(P)$, it holds that $[0,1]^n \subseteq (ax \geq a_0)$. We claim that the set $(ax \leq a_0) \cap [0,1]^n$ is a face of the unit cube: For this, assume w.l.o.g. that $(0,\ldots,0) \in E$ and therefore $a_0 = 0$ and $a_i \geq 0$, for $i = 1,\ldots,n$. Let I denote the set of indices i such that $a_i > 0$. Then

$$(ax = 0) \cap \{0,1\}^n = \{0,1\}^n \cap \left(\bigcap_{i \in I} (x_i = 0) \right)$$

implies that $(ax = 0) \cap [0,1]^n$ is, indeed, a face of the unit cube. Since the unit cube has only finitely many faces, there exists a finite set of M-cuts that dominate every M-cut in $M_{(L=\emptyset)}(P)$. Next, consider M-cuts $ax \leq a_0$ in sets $M_{(L,E,G)}(P)$ such that $G = \emptyset$. Then $[0,1]^n \subseteq (ax \leq a_0)$ and, hence, any such cut is dominated by the cube constraints. We define M_0 to be the intersection of the unit cube $[0,1]^n$ with the M-cuts in $M_{(L=\emptyset)}(P)$.

Let $\mathcal{P}$ denote the set of all partitions of $\{0,1\}^n$ into disjoint sets L, E, and G such that $L, E, G \neq \emptyset$. Then with the above observations and Lemma 5.3, it holds that

$$M(P) = M_0 \cap \bigcap_{(L,E,G) \in \mathcal{P}} \left\{ (\alpha x \leq \alpha_0) \,\middle|\, (\alpha, \alpha_0) \in Q_{(L,E,G)}(P) \cap \mathbb{Q}^{n+1} \right\} . \tag{5.9}$$

For any partition $(L,E,G) \in \mathcal{P}$, the polyhedron $Q_{(L,E,G)}(P)$ has a representation in terms of its finite set of vertices $\mathcal{V}_{(L,E,G)}$ and its finite set of extreme rays $\mathcal{R}(L,E,G)$ (see Theorem 2.1). That is,

$$Q_{(L,E,G)}(P) = \operatorname{conv}\left(\mathcal{V}(Q_{(L,E,G)})\right) + \operatorname{cone}\left(\mathcal{R}_{(L,E,G)}\right) . \tag{5.10}$$

Since P is rational, $Q_{(L,E,G)}(P)$ is a rational polyhedron and, hence, the points in $\mathcal{V}_{(L,E,G)}$ and $\mathcal{R}_{(L,E,G)}$ are rational. We claim that $M(P) = M^*(P) \cap M_0$, where

$$M^*(P) := \bigcap_{(L,E,G)\in\mathcal{P}} \left(\left\{ (vx \leq v_0) \,\middle|\, (v,v_0) \in \mathcal{V}_{(L,E,G)} \right\} \cap \left\{ (rx \leq r_0) \,\middle|\, (r,r_0) \in \mathcal{R}_{(L,E,G)} \right\} \right) \ . \tag{5.11}$$

Clearly, because of Lemma 5.3 and 5.4,

$$M(P) \subseteq M^*(P) \cap M_0 \ .$$

For the other inclusion, take any inequality $\alpha x \leq \alpha_0$ for $M(P)$ from representation (5.9) that is not valid for M_0, that is, $(\alpha,\alpha_0) \in Q_{(L,E,G)}(P) \cap \mathbb{Q}^{n+1}$. By (5.10), the point (α,α_0) can be written as a convex combination of solutions in $\mathcal{V}_{(L,E,G)}$ and nonnegative combinations of extreme rays in $\mathcal{R}_{(L,E,G)}$. Suppose that $\mathcal{V}_{(L,E,G)} = \{(v^1,v_0^1),\ldots,(v^s,v_0^s)\}$ and $\mathcal{R}_{(L,E,G)} = \{(r^1,r_0^1),\ldots,(r^t,r_0^t)\}$. Then there exist $\lambda_i \geq 0$, for $i = 1,\ldots,s$, and $\mu_j \geq 0$, for $j = 1,\ldots,t$, such that $\sum_{i=1}^s \lambda_i = 1$ and

$$(\alpha,\alpha_0) = \sum_{i=1}^{s} \lambda_i (v^i, v_0^i) + \sum_{j=1}^{t} \mu_j (r^j, r_0^j) \ .$$

For an an arbitrary point $x \in M^*(P)$, we have

$$\alpha\, x = \sum_{i=1}^{s} \lambda_i\, v^i x + \sum_{j=1}^{t} \mu_j\, r^j x \ \leq \ \sum_{i=1}^{s} \lambda_i\, v_0^i + \sum_{j=1}^{t} \mu_j\, r_0^j = \alpha_0 \ ,$$

that is, x satisfies the inequality $\alpha x \leq \alpha_0$. Hence, $M^*(P) \cap M_0 \subseteq M(P)$. Since the number of partitions of the set of 0/1 points in $\mathbb{R}^n$ is finite, that is, $|\mathcal{P}|$ is finite, $M^*(P)$ is a rational polyhedron. As a result, $M^*(P) \cap M_0$ is a polytope, implying that the same is true for $M(P)$. □

The proof of Theorem 5.5 does not extend to non-rational polytopes, since in that case, $Q_{(L,E,G)}(P)$ is not a rational polyhedron anymore. In particular, the sets $\mathcal{V}_{(L,E,G)}$ and $\mathcal{W}_{(L,E,G)}$ are no longer rational.

5.3.3 Facet Characterization

By definition (5.2), every facet-defining inequality for the M-closure of a non-empty polytope is tight at a 0/1 point. This is why in the proof of Theorem 5.5, we could restrict ourselves to partitions (L, E, G) of the set of 0/1 points with $E \neq \emptyset$. It turns out that the partitions that imply undominated M-cuts satisfy a stronger property: If $M(P)$ is a k-dimensional polytope and $ax \leq a_0$ a facet-defining M-cut for $M(P)$, the hyperplane $(ax = a_0)$ contains at least k affinely independent 0/1 points. Furthermore, the corresponding facet $F = P \cap (ax = a_0)$ is contained in the affine subspace spanned by the 0/1 point in $(ax = a_0)$. In the special case that P_I has full dimension, every facet-defining inequality for $M(P)$ is associated with a hyperplane that is spanned by n affinely independent 0/1 points and, therefore, corresponds to a facet of some 0/1 polytope.

Our proof of this property relies on the fact that the M-closure of a polytope is a polytope itself (see Section 5.3.2).

Theorem 5.6 *Let $P \subseteq [0,1]^n$ be a rational polytope and assume that the polytope $M(P)$ has dimension $k > 0$. If $ax \leq a_0$ is a facet-defining M-cut for $M(P)$, then $(ax = a_0)$ contains at least k affinely independent points in $\{0,1\}^n$. Furthermore,*

$$M(P) \cap (ax = a_0) \subseteq \operatorname{conv}\big((ax = a_0) \cap \{0,1\}^n\big) \ .$$

Proof. Let $ax \leq a_0$ be an M-cut for P that is facet-defining for $M(P)$. Let $F = M(P) \cap (ax = a_0)$ be the corresponding facet. Define

$$\begin{aligned} L &= \big\{x \in \{0,1\}^n \,\big|\, ax < a_0\big\} \ , \\ E &= \big\{x \in \{0,1\}^n \,\big|\, ax = a_0\big\} \ , \\ G &= \big\{x \in \{0,1\}^n \,\big|\, ax > a_0\big\} \ . \end{aligned}$$

Furthermore, let $S := \operatorname{conv}(E)$.

We will show that for any point $z \in (ax = a_0) \setminus S$, there exists an inequality $cx \leq c_0$ that is valid for $M(P)$ and violated by z, thereby proving that $F \subseteq S$. If we know that $F \subseteq S$, then $\dim(M(P)) = k$ implies that $k - 1 = \dim(F) \leq \dim(S)$. In particular, $(ax = a_0)$ must contain at least k affinely independent 0/1 points.

Consider an arbitrary point $z \in (ax = a_0) \setminus S$. The set S is a (lower-dimensional) integral polytope and can therefore be described by a finite system of rational inequalities. Since $z \notin S$, there exists some rational inequality $fx \leq f_0$ that is valid for S, but is violated by z; that is, $fz > f_0$. Suppose that we can find constants $\lambda > 0$ and $\varepsilon > 0$ such that $c := a + \lambda f$ and $c_0 := a_0 + \lambda f_0$ satisfy

(i) $cx \leq c_0$ for all $x \in L \cup E$,

(ii) $cx \leq \min\{cx \mid x \in G\} - \varepsilon$ is valid for P.

Then the inequality $cx \leq c_0$ is valid for $M(P)$, since the right-hand side of the M-cut for P associated with the valid inequality $cx \leq \min\{cx \mid x \in G\} - \varepsilon$ satisfies

$$\begin{aligned} KV\big(c, \{cx \mid x \in G\} - \varepsilon\big) &= \max\Big\{cx \,\Big|\, x \in \{0,1\}^n,\ cx \leq \min\{cx \mid x \in G\} - \varepsilon\Big\} \\ &= \max\Big\{cx \,\Big|\, x \in (L \cup E)\ cx \leq \min\{cx \mid x \in G\} - \varepsilon\Big\} \\ &\leq \max\Big\{cx \,\Big|\, x \in (L \cup E)\Big\} \leq c_0 \ . \end{aligned}$$

Furthermore, $cz = az + \lambda fz > a_0 + \lambda f_0 = c_0$, that is, $z \notin M(P)$.

In the remainder of the proof we will show that such λ and ε indeed exist. First, observe that property (i) holds for every point $x \in E \subseteq S$ for any choice of λ, since then $x \in (ax = a_0)$ and $x \in S$ imply that $cx = ax + \lambda fx \leq a_0 + \lambda f_0 = c_0$. Regarding the set L, let us define the constants $a_L := \max\{ax \mid x \in L\}$ and $f_L := \max\{fx \mid x \in L\}$. If

$$\lambda(f_L - f_0) \leq a_0 - a_L \ , \tag{5.12}$$

then we get for all $x \in L$ that $cx = ax + \lambda fx \leq a_L + \lambda f_L \leq a_0 + \lambda f_0 = c_0$. Note that $a_0 - a_L > 0$ and that condition (5.12) is trivially fulfilled for any $\lambda > 0$ if $f_L - f_0 \leq 0$. Now consider property (ii). Since $ax \leq a_0$ is an M-cut for P, we know that there exists some $\varepsilon_a > 0$ such that the inequality

$$ax \leq \min\{ax \mid x \in G\} - \varepsilon_a$$

is valid for P. If we choose λ and ε such that

$$\lambda \max\{fx \mid x \in P\} - \varepsilon_a \leq \lambda \min\{fx \mid x \in G\} - \varepsilon \ , \tag{5.13}$$

then for any $x \in P$,

$$\begin{aligned} cx = ax + \lambda fx &\le \min\{ax \mid x \in G\} - \varepsilon_a + \lambda \max\{fx \mid x \in P\} \\ &\le \min\{ax \mid x \in G\} + \lambda \min\{fx \mid x \in G\} - \varepsilon \\ &\le \min\{cx \mid x \in G\} - \varepsilon \ . \end{aligned}$$

Define $\varepsilon := \varepsilon_a/2 > 0$. Then condition (5.13) becomes

$$\lambda \left(\max\{fx \mid x \in P\} - \min\{fx \mid x \in G\} \right) \le \varepsilon_a/2 \ . \tag{5.14}$$

It is not difficult to see that for all possible four cases of signs of $f_L - f_0$ and $\max\{fx \mid x \in P\} - \min\{fx \mid x \in G\}$, we can find a $\lambda > 0$ such that conditions (5.12) and (5.14) are satisfied. Hence, there exist λ and ε such that properties (i) and (ii) hold. This completes the proof of the theorem. $\square$

In the case that P_I has full dimension, Theorem 5.6 represents a natural analogy to the fact that every hyperplane that is associated with a facet-defining Gomory-Chvátal cut is spanned by n affinely independent integral points. However, while this property of the Gomory-Chvátal closure is a direct consequence of its definition (only inequalities with integral normal vectors are considered for the computation of the Gomory-Chvátal closure), the corresponding property for the M-closure does not immediately follow from Definition 5.2. Observe, furthermore, that in the above proof we specifically used the fact that F can be described as the intersection of $M(P)$ with a hyperplane ($ax = a_0$) that corresponds to an M-cut. Theorem 5.6 does not apply to arbitrary faces of $M(P)$. In particular, there are face-defining inequalities for which the associated hyperplanes do not contain any 0/1 points (see Figure 5-4).

5.4 Comparison of the M-Closure with other Elementary Closures of 0/1 Integer Programs

Numerous families of cutting planes and their associated elementary closures have been introduced and studied in the literature. Surprisingly, several of these families have

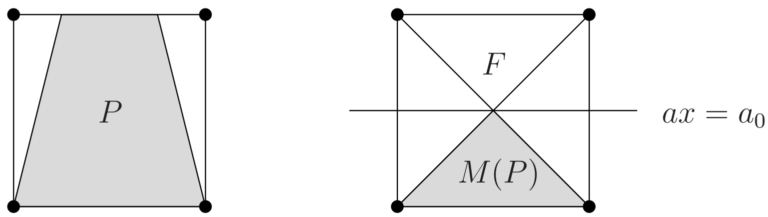

Figure 5-4: The inequality $ax \leq a_0$ is face-defining for $M(P)$, but the hyperplane $(ax = a_0)$ does not contain any 0/1 points.

turned out to be essentially identical, even though they had been derived by very different methods. For example, Cornuéjols and Li (2000) showed that Gomory-Chvátal cuts and *Gomory fractional cuts* are equivalent, even though the latter are derived from an equality system describing the polytope $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$. Similarly, the closures associated with *mixed integer rounding cuts*, *disjunctive cuts*, and *Gomory mixed integer cuts* are identical. In particular, any disjunctive cut can be obtained from the mixed integer rounding procedure, Gomory mixed integer cuts can be derived from a disjunction expressing integrality conditions on the x-variables, and mixed integer rounding cuts can be derived by Gomory's mixed integer procedure (see Cornuéjols and Li 2000 for definitions of families of cutting planes for 0/1 integer programs and a detailed comparison of their elementary closures). Figure 5-5 illustrates various known closures and their pairwise relationships. In this section, we investigate how the M-closure fits into this picture. Specifically, we compare the M-closure of a polytope with each of the closures depicted in Figure 5-5.

As demonstrated in Theorem 5.6, the hyperplanes in $\mathbb{R}^n$ that are spanned by n affinely independent 0/1 points are of fundamental importance for the M-closure of a polytope. Therefore, we will introduce a special notation for the M-cuts that define hyperplanes with this property. We will denote by $\mathcal{F}^n$ the set of pairs $(a, a_0) \in \mathbb{Z}^{n+1}$ with $\gcd(a) = 1$ for which the hyperplane $(ax = a_0)$ is spanned by n affinely independent points in $\{0,1\}^n$. Thus, for any facet-defining inequality $ax \leq a_0$ of an arbitrary 0/1 polytope, the pair (a, a_0) is contained in $\mathcal{F}^n$. Conversely, every pair $(a, a_0) \in \mathcal{F}^n$ defines a facet of some full-dimensional 0/1 polytope. When we say in the sequel that an inequality $ax \leq a_0$ is an inequality in $\mathcal{F}^n$, we refer to the fact that $(a, a_0) \in \mathcal{F}^n$.

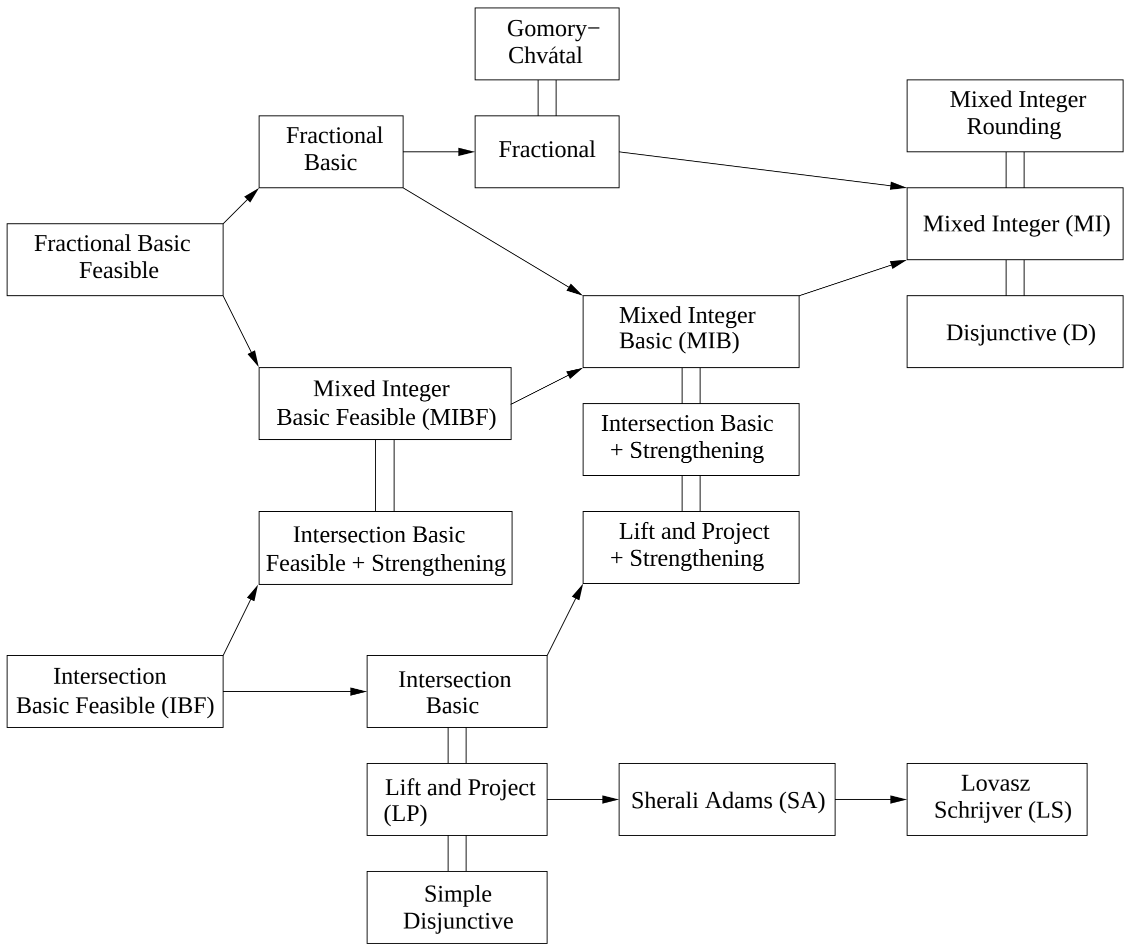

Figure 5-5: Comparison of elementary closures of 0/1 integer programs (taken from Cornuéjols and Li 2000). An arrow from one family of cuts F_1 to another family of cuts F_2 indicates that the closure P_{F_2} is contained in the closure P_{F_1}, and there exist instances for which the inclusion is strict. If two families F_1 and F_2 are unrelated, there exist instances such that $P_{F_1} \not\subseteq P_{F_2}$ and instances such that $P_{F_2} \not\subseteq P_{F_1}$.

5.4.1 Comparison of the M-Closure with the Gomory-Chvátal Closure

The close relationship between the M-closure and the classic Gomory-Chvátal cutting plane procedure is self-evident. We introduced the M-closure as a natural strengthening of the Gomory-Chvátal closure for the special case that all integer points of a polyhedron are 0/1 points. By definition, every Gomory-Chvátal cut of a polytope P defines a valid inequality for its M-closure and, hence, $M(P) \subseteq P'$. On the other hand, an M-cut can strictly dominate the corresponding Gomory-Chvátal cutting plane. If $ax \leq a_0$ is a Gomory-Chvátal cut for P and facet-defining for P' and if the hyperplane $(ax = a_0)$ does not contain a 0/1 point, the right-hand side associated with the M-cut is strictly smaller than a_0, implying that $M(P) \subset P'$. Conversely, one can ask the question of under which circumstances a facet-defining M-cut is also a Gomory-Chvátal cut. For this, consider a polytope P with full-dimensional integer hull. By Theorem 5.6, every facet F of $M(P)$ is contained in an affine space that is spanned by 0/1 points. Put differently, $F \subseteq (ax = a_0)$ for some pair $(a, a_0) \in \mathcal{F}^n$. If for every inequality $ax \leq a_0$ in $\mathcal{F}^n$ there was a 0/1 point in the hyperplane $(ax = a_0 + 1)$, any such inequality $ax \leq a_0$ would be valid for $M(P)$ if and only if it was also a Gomory-Chvátal cut. As a consequence, it would hold that $M(P) = P'$ for every polytope P with full-dimensional integer hull. On the other hand, if we were to find an inequality $ax \leq a_0$ in $\mathcal{F}^n$ for which $(ax = a_0 + 1) \cap \{0,1\}^n = \emptyset$, this would immediately enable us to construct a polytope P for which the M-closure $M(P)$ is strictly contained in its elementary closure P' (see Figure 5-6 for an illustration).

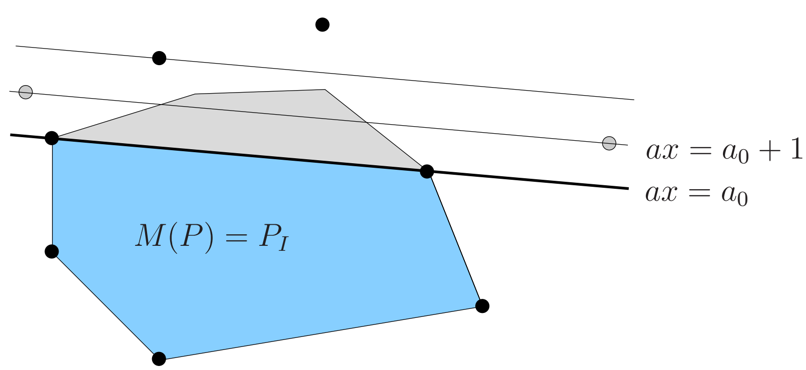

Figure 5-6: The hyperplane $(ax = a_0)$ with $\gcd(a) = 1$ is spanned by two 0/1 points and, therefore, facet-defining for some 0/1 polytope. If the hyperplane $(ax = a_0 + 1)$ does not contain a 0/1 point, we can construct a polytope P such that $(ax \leq a_0)$ is facet-defining for P_I and such that $M(P) \subset P'$. (Note that points in $\{0,1\}^n$ are illustrated as black points and general integer points are drawn in grey.)

The following lemma summarizes these observations.

Lemma 5.7 *$M(P) = P'$ for every rational polytope $P \subseteq [0,1]^n$ with full-dimensional integer hull P_I if and only if $(ax = a_0 + 1) \cap \{0,1\}^n \neq \emptyset$ for every inequality $ax \leq a_0$ in $\mathcal{F}^n$.*

The characterization in Lemma 5.7 allows us to compare the Gomory-Chvátal closure and the M-closure of any rational polytope with full-dimensional integer hull in small dimension. It is possible to simply enumerate all inequalities $ax \leq a_0$ in $\mathcal{F}^n$ and check whether $(ax = a_0 + 1) \cap \{0,1\}^n \neq \emptyset$. It turns out that this is indeed the case for $n \leq 7$. That is, for every pair $(a, a_0) \in \mathcal{F}^n$, the hyperplane $(ax = a_0 + 1)$ contains a 0/1 point.

Corollary 5.8 *$M(P) = P'$ for every rational polytope $P \subseteq [0,1]^n$ with full-dimensional integer hull, if $n \leq 7$.*

For complexity reasons, it was not possible to enumerate the inequalities in $\mathcal{F}^n$ for $n > 7$. The above condition, therefore, does not allow us to draw any general conclusions for higher dimensions. However, the following example confirms the general intuition that there should exist polytopes of some dimension for which the M-closure is strictly contained in the Gomory-Chvátal closure.

Example 5.9 *The integral vector $a \in \mathbb{R}^{31}$ with*

$$\begin{aligned} a \; := \; (&3041115360, & 3216174509, && 3081631465, && 2598051222, && 2888963817, \\ &-129947214, & 25001283, && -76282297, && -384543308, && 24697928, \\ &-102658009, & -900040667, && 157047656, && -207693535, && -883266807, \\ &-740003674, & -352003226, && 458140458, && -261010564, && 566248994, \\ &-360665679, & -185629660, && -305123253, && -182278292, && -272079117, \\ &-518683, & 311565566, && -569860449, && -691996681, && -303576202, \\ &1885519442) \end{aligned}$$

has relatively prime components. The hyperplane $(ax = 0)$ is spanned by affinely independent points in $\{0,1\}^{31}$, that is, $(a, 0) \in \mathcal{F}^{31}$. Furthermore, $(ax = 1) \cap \{0,1\}^n = \emptyset$.

(The smallest right-hand side $a_0 > 0$ such that $ax = a_0$ contains a 0/1 point is $a_0 = 8$.)

The above example has not been chosen at random. Joswig published a list of 0/1 polytopes that have facet-defining inequalities with very large integer coefficients. These polytopes were generated based on a construction of 0/1 matrices with large determinants by Alon and Vu (1997). Geometrically, the hyperplanes described by 0/1 matrices with large determinants slice the unit cube non-symmetrically. They can be thought of as the most skewed hyperplanes spanned by 0/1 points. The inequality $ax \leq 0$ from Example 5.9 is facet-defining for one of these 0/1 polytopes (named "MJ:32-33"). Interestingly, we were unable to generate an example with the same property randomly. For every hyperplane H that we generated by randomly picking n affinely independent 0/1 points (for dimensions up to 35), the parallel hyperplanes obtained by shifting H to the next level of integer points above and below the hyperplane also contained a 0/1 point. This might suggest that the number of hyperplanes $(ax = a_0)$ that are spanned by 0/1 points and that satisfy $(ax = a_0 + 1) \cap \{0,1\}^n = \emptyset$ is small compared to the size of $\mathcal{F}^n$.

Example 5.9 allows us to specifically construct polytopes in arbitrary dimension for which the M-closure is strictly contained in the Gomory-Chvátal closure.

Corollary 5.10 *For every rational polytope $P \subseteq [0,1]^n$, $M(P) \subseteq P'$. Furthermore, there exists a number $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ there exists a polytope $P \subseteq [0,1]^n$ with full-dimensional integer hull such that $M(P) \subset P'$.*

Proof. Since every M-cut associated with a valid inequality for P dominates the corresponding Gomory-Chvátal cut, we have $M(P) \subseteq P'$. For the second part of the corollary, we construct an example: Let $n = 31$ and $P := [0,1]^n \cap (ax \leq 7)$, where a is the normal vector defined in Example 5.9. Since $(a, 0) \in \mathcal{F}^n$ and $(ax > 0) \cap (ax \leq 7) \cap \{0,1\}^n = \emptyset$, the inequality $ax \leq 0$ must be facet-defining for P_I. In particular, it cannot be implied by other inequalities that are valid for P_I. Clearly, $ax \leq 0$ is valid for $M(P)$. However, the inequality is not a Gomory-Chvátal cut and, since it is facet-defining for P_I, it cannot be implied by other Gomory-Chvátal cuts, that is, $M(P) \subset P'$. Furthermore, P_I is full-dimensional, since the 0/1 points that span the hyperplane $(ax = 0)$ together with the unit vector $e_6 \in \{0,1\}^{31}$ are $n+1$ affinely independent points (note that $a\, e_6 = a_6 < 0$).

For every $n > n_0 := 31$, we can extend a to an n-dimensional vector by adding $n - n_0$ zero components and the same argument applies. $\square$

5.4.2 Comparison of the M-Closure with the Knapsack Closure

Another elementary closure that is very closely related to the M-closure is the *knapsack closure*, which was formally introduced by Fischetti and Lodi (2010) for polytopes in the unit cube. The definition of the knapsack closure is inspired by an observation of Crowder, Johnson, and Padberg (1983): if $P \subseteq [0,1]^n$ is defined by inequalities $a_i x \leq b_i$, for $i = 1, \ldots, m$, then

$$P_I \subseteq \bigcap_{i=1}^{m} P_I^i \ ,$$

where $P_I^i = \operatorname{conv}\{x \in \{0,1\}^n \,|\, a_i x \leq b_i\}$. That is, P_I is contained in the intersection of all knapsack polytopes defined by the constraints of P. As the example in Figure 5-7 illustrates, it is possible that the intersection of these knapsack polytopes results in P itself, that is, it does not provide a better approximation of the integer hull.

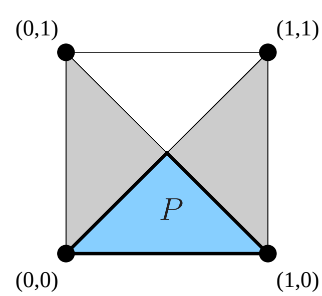

Figure 5-7: The polytope $P \subseteq [0,1]^2$ is defined by the inequalities $-x_1 + x_2 \leq 0$, $x_1 + x_2 \leq 1$, and $x_2 \geq 0$ and $P_I = \operatorname{conv}\{(0,0);(1,0)\}$. Since each constraint defines an integral half-space, $P_I^1 \cap P_I^2 \cap P_I^3 = P$.

However, if one intersects the knapsack polytopes associated with *all* valid inequality for P, the resulting set certainly represents a better approximation of the integer hull P_I.

The knapsack closure of a polytope $P \subseteq [0,1]^n$ is defined as

$$K(P) := \bigcap_{a \in \mathbb{Z}^n} \mathrm{conv}\big\{x \in \{0,1\}^n \,\big|\, ax \leq a_P\big\} \ . \tag{5.15}$$

As the knapsack closure clearly dominates the Gomory-Chvátal closure, its iterative application to a polytope will generate its integer hull a finite number of steps. Furthermore, it is easy to see that the knapsack closure of a rational polytope is a polytope itself. (In fact, the knapsack closure of an arbitrary set in $[0,1]^n$ is a polytope.)

Lemma 5.11 *For any polytope $P \subseteq [0,1]^n$, the knapsack closure $K(P)$ is a rational polytope.*

Proof. The knapsack closure is an infinite intersection of knapsack polytopes. However, only a finite number of different knapsack polytopes in $\mathbb{R}^n$ exist, since every such polytope is uniquely defined by the set of 0/1 points that violate the knapsack inequality. More precisely, the number of different knapsack polytopes is given by the number of possible subsets $S \subseteq \{0,1\}^n$ such that S can be separated from $\{0,1\}^n \setminus S$ by a hyperplane. □

We can characterize the facets of the knapsack closure of a polytope in the following way:

Lemma 5.12 *Let $P \subseteq [0,1]^n$ be a polytope with full-dimensional integer hull. If $ax \leq a_0$ is a facet-defining inequality for $K(P)$ with $(a, a_0) \in \mathbb{Z}^{n+1}$ and $\gcd(a) = 1$, then the following properties hold:*

(i) $ax \leq a_0$ *is in* $\mathcal{F}^n$.

(ii) There exists a valid inequality $cx \leq c_0$ for P such that

$$(cx \leq c_0) \cap \{0,1\}^n \subseteq (ax \leq a_0) \cap \{0,1\}^n \ .$$

Proof. Property (i) follows from the definition of the knapsack closure as the intersection of knapsack polytopes: Since P_I is full-dimensional, every knapsack polytope containing P is full-dimensional. Therefore, every facet of $K(P)$ is a facet of some full-dimensional 0/1 polytope, that is, it contains n affinely independent 0/1 points. Now consider property (ii): If $ax \leq a_0$ defines a facet of $K(P)$, there must exist a valid inequality $cx \leq c_0$ for P such that $ax \leq a_0$ is a facet of $\operatorname{conv}\{x \in \{0,1\}^n \mid cx \leq c_0\}$. Since $\operatorname{conv}\{x \in \{0,1\}^n \mid cx \leq c_0\} \subseteq (ax \leq a_0)$, we have $(cx \leq c_0) \cap \{0,1\}^n \subseteq (ax \leq a_0)$ and (ii) follows. □

As shown in Theorem 5.6, property (i) of Lemma 5.12 also applies to the M-closure of P. Property (ii) provides that for every facet-defining inequality $ax \leq a_0$ of $K(P)$ there must be a valid inequality $cx \leq c_0$ for P that is violated by at least the 0/1 points in $(ax > a_0)$. This property is also true for the M-closure, since we can choose $cx \leq c_0$ as $ax \leq \min\{ax \mid x \in \{0,1\}, ax > a_0\} - \varepsilon$, for some $\varepsilon > 0$. However, there is a significant difference between the M-closure and the knapsack closure:

Corollary 5.13 *Let $P \subseteq [0,1]^n$ be a polytope and $S \subseteq \{0,1\}^n$. If there exists a valid inequality $cx \leq c_0$ for P with $(cx > c_0) \cap \{0,1\}^n = S$, then any inequality $ax \leq a_0$ with $(ax > a_0) \cap \{0,1\}^n \subseteq S$ is valid for $K(P)$.*

Proof. The inequality $ax \leq a_0$ is a valid for $\operatorname{conv}\{x \in \{0,1\}^n \mid cx \leq c_0\}$. □

It is not immediately obvious whether the above property also holds for the M-closure. If it did, every cut that is valid for $K(P)$ would also be valid for $M(P)$ and, hence, the two closures would be identical. Equivalently, the property would imply that the M-closure of any polytope defined by a single constraint (in addition to the cube constraints) would give its integer hull. That is, for any polytope $P = \{x \in [0,1]^n \mid cx \leq c_0\}$, it would hold that $M(P) = P_I$. The following example shows that this is, in general, not true.

Example 5.14 *For $n = 7$, let $cx \leq c_0$ denote the inequality*

$$-3x_1 - 6x_2 + 7x_3 + 3x_4 + x_5 + x_6 + 2x_7 \leq 5 \ ,$$

and let $P = \{x \in [0,1]^n \mid cx \leq c_0\}$. One can verify that $ax \leq a_0$ given by

$$-3x_1 - 2x_2 + 3x_3 + 3x_4 + x_5 + x_6 + 2x_7 \leq 5$$

is facet-defining for P_I*. Furthermore, it holds that*

$$(cx \leq c_0) \cap \{0,1\}^n \subseteq (ax \leq a_0) \cap \{0,1\}^n \ .$$

However, $\max\{ax \mid x \in P\} > 6$*, and* $(ax = 6) \cap \{0,1\}^n \neq \emptyset$*. Therefore,* $ax \leq a_0$ *is not an M-cut for* P*, implying* $M(P) \supset K(P)$*.*

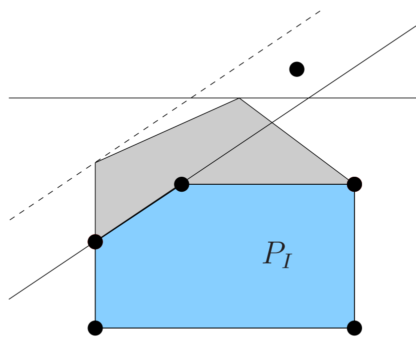

Figure 5-8: Schematic illustration of the difference between the M-closure and the knapsack closure: The inequality $ax \leq a_0$ is valid for $K(P)$, but not for $M(P)$. The inequality $cx \leq c_0$ is valid for P and cuts off the same set of 0/1 points as $ax \leq a_0$. (Note that only integer points in $\{0,1\}^n$ are illustrated.)

Example 5.14 shows that Corollary 5.13 does not apply to the M-closure. It also highlights a key difference between the M-closure and the knapsack closure of a polytope P: In order for an inequality $ax \leq a_0$ in $\mathcal{F}^n$ to be valid for $K(P)$, it is sufficient that there exists some valid inequality $cx \leq c_0$ for P that cuts off the same set S of 0/1 points as $ax \leq a_0$, or a larger set. In contrast, for $ax \leq a_0$ to be a valid for the M-closure, there must exist a valid inequality for P that separates the points in S in a specific way. Intuitively, the angle between the normal vectors a and c and, therefore, the angle between the associated hyperplanes, must not be arbitrarily large (see Figure 5-8).

Corollary 5.15 *For every polytope* $P \subseteq [0,1]^n$*,* $K(P) \subseteq M(P)$*. Furthermore, there exists a number* $n_0 \in N$ *such that for every* $n \geq n_0$ *there is a polytope* $P \subseteq [0,1]^n$ *with full-dimensional integer hull such that* $K(P) \subset M(P)$*.*

The following figure summarizes the observations of Sections 5.4.1 and 5.4.2.

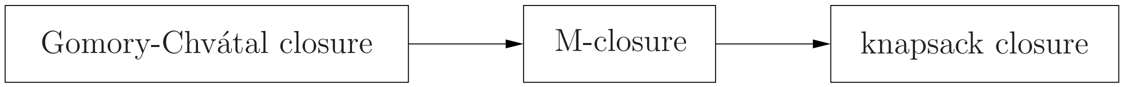

Figure 5-9: Relationship between the Gomory-Chvátal closure, the M-closure, and the knapsack closure of a rational polytope.

5.4.3 Comparison of the M-Closure with Elementary Closures Derived from Fractional Cuts, Lift-and-Project Cuts, Intersection Cuts, and Disjunctive Cuts

Cornuéjols and Li (2000) showed that there is no universal relationship between the Gomory-Chvátal closure and most of the elementary closures depicted in Figure 5-5. More precisely, they gave examples of polytopes for which the Gomory-Chvátal closure is not contained in the other elementary closures and, in turn, showed the reverse for other instances.

Example 5.16 (Cornuéjols and Li 2000) *Consider the 2-dimensional polytope*

$$P = \left\{ x \in \mathbb{R}^2 \,\middle|\, -2x_1 + x_2 \leq 0;\ 2x_1 + x_2 \leq 2;\ x_2 \geq 0 \right\} \tag{5.16}$$

with $P_I = \text{conv}\{(0,0);(1,0)\} = \{x \in \mathbb{R}^2 \mid 0 \leq x_1 \leq 1;\ -x_2 \leq 0;\ x_2 \leq 0\}$ *(see Figure 5-10 for an illustration). The inequality* $x_2 \leq 0$ *is an intersection cut derived from the basic feasible solution* $(1/2, 1)$*. Therefore,* $P_{IBF} = P_I$*, that is, the elementary closure with respect to intersection cuts derived from basic feasible solutions is equal to the integer hull of* P*. However, every Gomory-Chvátal cut is satisfied by the point* $(1/2, 1/2)$*, implying* $P' \neq P_I$*. Thus,* $P' \not\subseteq P_{IBF}$*.*

As can be seen in Figure 5-5, the elementary closure P_{IBF} of P associated with the family of intersection cuts derived from basic feasible solutions is the weakest relax-

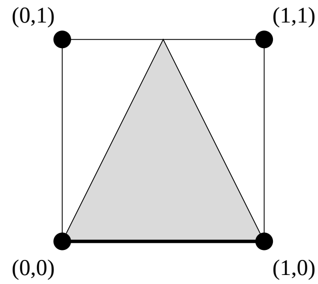

Figure 5-10: Example of a polytope $P \subseteq [0,1]^n$ for which the Gomory-Chvátal closure is not contained in the elementary closure with respect to intersection cuts derived from all basic feasible solutions, that is, $P' \not\subseteq P_{IBF}$.

ation of the integer hull P_I among $P_{IBF}, P_{MIBF}, P_{LP}, P_{SA}, P_{LS}, P_{MIB}$, and P_{MI}. Consequently, Example 5.16 establishes for all these closures the existence of instances of polytopes such that the Gomory-Chvátal closure is not contained in them. Moreover, since $K(P) = M(P) = P'$ for the polytope defined by (5.16), Example 5.16 establishes the same result for the M-closure $M(P)$ and the knapsack closure $K(P)$.

Corollary 5.17 *For every $n \geq 2$, there exist instances of full-dimensional polytopes $P \subseteq [0,1]^n$, such that $M(P) = K(P) \not\subseteq P_{IBF}, P_{MIBF}, P_{LP}, P_{SA}, P_{LS}, P_{MIB}, P_{MI}$.*

Cornuéjols and Li (2000) also provided examples of polytopes for which $P_{MIB} \not\subseteq P'$ and polytopes such that $P_{LS} \not\subseteq P_{FBF}$. Since $K(P) \subseteq M(P) \subseteq P' \subseteq P_{FBF}$ and because of the relationships between the closures depicted in Figure 5-5, the following observation is obtained.

Corollary 5.18 *For every $n \geq 2$, there exist instances of polytopes $P \subseteq [0,1]^n$ such that $P_{IBF}, P_{MIBF}, P_{LP}, P_{SA}, P_{LS}, P_{MIB} \not\subseteq M(P) \supseteq K(P)$.*

So far, we have established the incomparability between the M-closure and knapsack closure and $P_{IBF}, P_{MIBF}, P_{LP}, P_{SA}, P_{LS}, P_{MIB}$ (and all closures equivalent to these). For mixed integer cuts or, equivalently, disjunctive cuts, we know only from Corollary 5.17 that there are instances such that $K(P) = M(P) \not\subseteq P_{MI} = P_D$, but no inferences for the other direction can be drawn. However, in the next lemma, we construct an example for the reverse inclusion.

Lemma 5.19 *For every $n \geq 7$, there exist instances of polytopes $P \subseteq [0,1]^n$ such that $P_D = P_{MI} \nsubseteq M(P)$, and thus also $P_D = P_{MI} \nsubseteq K(P)$.*

Proof. Let $n := 31$ and $P := [0,1]^n \cap (ax \leq 7)$, where a is the normal vector defined in Example 5.9. As seen above, $ax \leq 0$ is an M-cut for P and facet-defining for P_I. We will show that $ax \leq 0$ is not valid for P_D. Suppose there exists a pair $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ such that the corresponding disjunction implies $ax \leq 0$. That is, $P_1 := P \cap (\pi x \leq \pi_0) \subseteq (ax \leq 0)$ and $P_2 := P \cap (\pi x \geq \pi_0 + 1) \subseteq (ax \leq 0)$. If $P_1 \neq \emptyset$ and $P_2 \neq \emptyset$, then

$$\begin{array}{lcl} \max\left\{ax \,\middle|\, x \in [0,1]^n,\ ax \leq 7,\ \pi x \leq \pi_0\right\} & \leq & 0 \ , \\ \max\left\{ax \,\middle|\, x \in [0,1]^n,\ ax \leq 7,\ \pi x \geq \pi_0 + 1\right\} & \leq & 0 \end{array}$$

implies

$$\begin{array}{lcl} \max\left\{ax \,\middle|\, x \in [0,1]^n,\ \pi x \leq \pi_0\right\} & \leq & 0 \ , \\ \max\left\{ax \,\middle|\, x \in [0,1]^n,\ \pi x \geq \pi_0 + 1\right\} & \leq & 0 \ . \end{array}$$

Hence, both $\pi x \leq \pi_0$ and $\pi x \geq \pi_0 + 1$ have to dominate $ax \leq 0$ over the unit cube, which is not possible. Therefore, let us assume w.l.o.g. that $P_2 = \emptyset$. Then $ax \leq 7$ implies $\pi x < \pi_0 + 1$ for any point $x \in [0,1]^n$ and we get that every $x \in P \cap \{0,1\}^n$ satisfies $\pi x \leq \pi_0$. Therefore, $\pi x \leq \pi_0$ is valid for P_I and $P_1 \neq \emptyset$. In particular, every 0/1 point in $(ax = a_0)$ satisfies $\pi x \leq \pi_0$. Since $P_1 \subseteq (ax \leq 0)$ by assumption, $\pi x \leq \pi_0$ has to dominate $ax \leq 0$ over the unit cube which is, given the other observations made above, only possible if $\pi x = \pi_0$ contains the same 0/1 points as $(ax = 0)$. This is only possible if $\pi x \leq \pi_0$ and $ax \leq a_0$ define the same half-space. But then $P_2 = P \cap (\pi x \geq \pi_0 + 1) \supseteq P \cap (ax \geq 1) \neq \emptyset$, which is a contradiction. □

Figure 5-11 summarizes the results of this subsection.

5.5 Bounds on the M-Rank

The M-closure of a rational polytope $P \subseteq [0,1]^n$ defines a tighter approximation of its integer hull P_I than P itself, assuming $P \neq P_I$. More precisely, it is at least as tight as

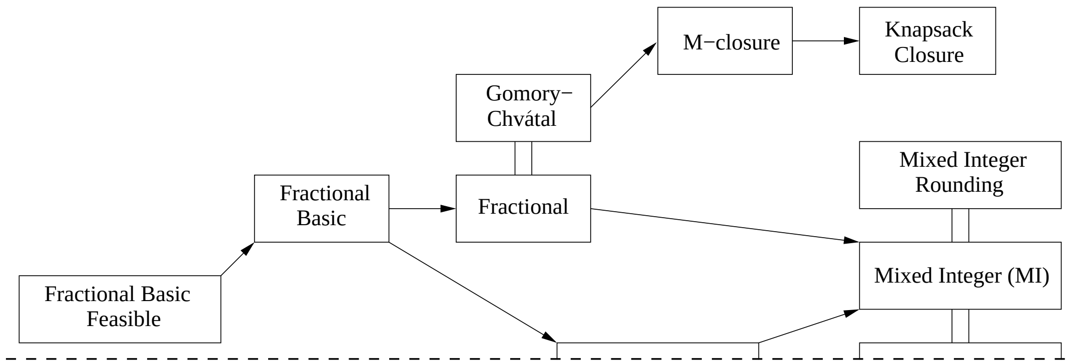

Figure 5-11: Placement of the M-closure and the knapsack closure into Figure 5-5.

the Gomory-Chvátal closure of the polytope and in certain cases it strictly dominates P'. Therefore, it is interesting to compare the sequences of successively tighter relaxations that arise from an iterative application of the M-closure operation with the sequences obtained during the Gomory-Chvátal procedure.

Clearly, as $M(P) \subseteq P'$, the M-rank cannot exceed the Chvátal rank of a polytope. Hence, by Theorem 2.19, we can conclude that for any polytope in the unit cube the M-rank is polynomially bounded in the dimension n.

Corollary 5.20 *If $P \subseteq [0,1]^n$ is a rational polytope, then the M-rank of P is bounded by a function in $O(n^2 \log n)$.*

Since we know that there are polytopes for which the M-closure is strictly contained in the Gomory-Chvátal closure, this raises the question of whether a better general upper bound for the M-rank can be obtained. This question is also of interest in light of the fact that the upper bound of Theorem 2.19 for the Chvátal rank has not been shown to be tight, that is, there is a significant gap between this bound and the worst-case example that has been exhibited so far. Unfortunately, we were unable to improve the upper bound on the M-rank given in Corollary 5.20. This question, therefore, remains the main open problem of this thesis and we will discuss it further in Chapter 6. However, we will show in the next subsection that, in the special case of a polytope with empty integer hull, the worst-case bound that holds for the Chvátal rank also applies for the M-rank.

5.5.1 Upper Bounds for Polytopes in the Unit Cube without Integral Points

The Chvátal rank of a polytope $P \subseteq [0,1]^n$ without integral points does not exceed n (Bockmayr et al. 1999) and this bound is tight.

Theorem 5.21 *Let P be a d-dimensional rational polytope in $[0,1]^n$ with $P_I = \emptyset$. If $d = 0$, then $P' = \emptyset$. If $d > 0$, then $P^{(d)} = \emptyset$.*

A family of polytopes for which exactly n iterations of the Gomory-Chvátal procedure are necessary in order to obtain an empty polytope is given by

$$P_n = \left\{ x \in [0,1]^n \;\middle|\; \sum_{i \in I} x_i + \sum_{i \notin I} (1 - x_i) \geq \frac{1}{2},\; I \subseteq \{1, \ldots, n\} \right\} . \tag{5.17}$$

P_n is the convex hull of all midpoints of the edges of the unit cube (see Figure 5-12). To see that the Chvátal rank of P_n is n, let F_j denote the set of all vectors in $\mathbb{R}^n$

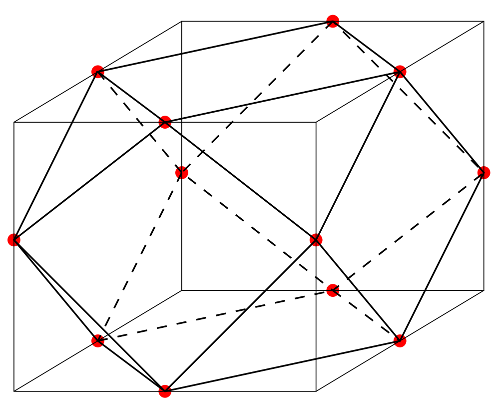

Figure 5-12: A family of polytopes in the n-dimensional unit cube with empty integer hull and M-rank n.

such that j components are $1/2$ and each of the remaining components is either 0 or 1. Chvátal, Cook, and Hartmann (1989) showed that, if a polytope P contains the set F_j, then F_{j+1} must be contained in its Gomory-Chvátal closure P'. Since it holds that $F_1 \subseteq P_n$, it follows that $F_n \subseteq P_n^{(n-1)}$. Hence, the Chvátal rank of P_n is at least n. With the upper bound from Theorem 5.21, the rank must be precisely n.

As we show next, the same family of polytopes also requires n iterations of the M-procedure to obtain the empty integer hull.

Lemma 5.22 *Let P be a d-dimensional rational polytope in $[0,1]^n$ with $P_I = \emptyset$. If $d = 0$, then $M(P) = \emptyset$. If $d > 0$, then $M^{(d)}(P) = \emptyset$. Furthermore, for every $n \geq 1$, there exists a rational polytope $P \subseteq [0,1]^n$ with $P_I = \emptyset$ and M-rank n.*

Proof. The first part of the lemma follows directly from Theorem 5.21 and the fact that $M(P) \subseteq P'$. In order to establish the tightness of this bound, we first show that if P contains F_j, then $M(P)$ contains F_{j+1}, for all $j = 1, \dots, n-1$.

Let $(a, a_0) \in \mathbb{Z}^{n+1}$ such that $ax \leq a_0$ is a valid inequality for P. As the set F_j is symmetric with respect to permutations of the coordinates and flipping signs of coordinates, we can assume w.l.o.g. $a_1 \geq a_2 \geq \dots \geq a_n \geq 0$. Suppose that F_j is contained in $(ax \leq a_0)$. We need to show that

$$F_{j+1} \subseteq \Big(ax \leq \max\big\{ax \,\big|\, x \in \{0,1\}^n, ax \leq a_0\big\}\Big) .$$

If j is even, then

$$\begin{aligned}
\max\{a\,y \mid y \in F_{j+1}\} &= a_1 + \dots + a_{n-(j+1)} + 1/2\;(a_{n-j} + \dots + a_n) \\
&\leq a_1 + \dots + a_{n-(j+1)} + 1/2\;(2a_{n-j} + 2a_{n-j+2} + \dots + 2a_{n-2} + a_n) \\
&\leq a_1 + \dots + a_{n-(j+1)} + (a_{n-j} + a_{n-j+2} + \dots + a_{n-2}) + a_n \\
&= a_1 + \dots + a_{n-j} + 1/2\;(2a_{n-j+2} + \dots + 2a_{n-2} + 2a_n) \\
&\leq a_1 + \dots + a_{n-j} + 1/2\;(a_{n-j+1} + \dots + a_n) \\
&= \max\{a\,y \mid y \in F_j\} \;.
\end{aligned}$$

If j is odd, then

$$\begin{aligned}\max\{a\,y \mid y \in F_{j+1}\} &= a_1 + \ldots + a_{n-(j+1)} + 1/2\ (a_{n-j} + \ldots + a_n) \\ &\le a_1 + \ldots + a_{n-(j+1)} + 1/2\ (2a_{n-j} + 2a_{n-j+2} + \ldots + 2a_{n-1}) \\ &= a_1 + \ldots + a_{n-(j+1)} + (a_{n-j} + a_{n-j+2} + \ldots + a_{n-1}) \\ &\le a_1 + \ldots + a_{n-j} + 1/2\ (2a_{n-j+2} + \ldots + 2a_{n-1}) + 1/2\ a_n \\ &\le a_1 + \ldots + a_{n-j} + 1/2\ (a_{n-j+1} + \ldots + a_n) \\ &= \max\{a\,y \mid y \in F_j\}\ .\end{aligned}$$

In both cases, the sequences of inequalities contain a line in which all coefficients of the a_i are either 0 or 1. Therefore, there always exists some $\bar{x} \in \{0,1\}^n$ such that

$$\max\{a\,y \mid y \in F_{j+1}\} \le a\bar{x} \le \max\{a\,y \mid y \in F_j\}\ .$$

Since $\max\{a\,y \mid y \in F_j\} \le a_0$, we get for all $y \in F_{j+1}$,

$$a\,y \le \max\{ax \mid x \in \{0,1\}^n, ax \le a_0\}\ .$$

As $F_1 \subseteq P_n$, the above observations imply that $F_n \subseteq M^{(n-1)}(P_n)$ and, hence, the M-rank of P_n is at least n. Since the M-rank cannot exceed n, it is exactly n. □

We want to mention here that the cutting plane procedure associated with the family of knapsack cuts (see Section 5.4.2) has the same worst-case bound, that is, not more than n successive applications of the knapsack closure operation are necessary for polytopes with empty integer hull to obtain P_I. Again, the family of polytopes defined by (5.17) serves as an example for which this bound is achieved.

Lemma 5.23 *Let P be a d-dimensional rational polytope in $[0,1]^n$ with $P_I = \emptyset$. If $d = 0$, then $K(P) = \emptyset$. If $d > 0$, then $K^{(d)}(P) = \emptyset$. Furthermore, for every $n \ge 1$, there exists a rational polytope P with $P_I = \emptyset$ such that n applications of the knapsack-closure operation are necessary to obtain P_I.*

Proof. As in the proof of Lemma 5.22, we will show that, if a polytope P contains F_j, then $K(P)$ contains F_{j+1}, for all $j = 1, \dots, n-1$. For this, it suffices to show that for any inequality $ax \le a_0$ such that $F_j \subseteq (ax \le a_0)$, it holds that

$$F_{j+1} \subseteq \operatorname{conv}\{x \in \{0,1\}^n \mid ax \le a_0\} \ .$$

Let $(a, a_0) \in \mathbb{Z}^{n+1}$, such that $F_j \subseteq (ax \le a_0)$. As F_j is invariant to coordinate flips, we can assume w.l.o.g. that $a \ge 0$. Consider an arbitrary $y \in F_{j+1}$. By renaming the indices, we can assume w.l.o.g. that $y_i = 1/2$ for $i = 1, \dots, j+1$, $y_i \in \{0,1\}$ for $i = j+2, \dots, n$, and $a_1 \le a_2 \le \dots \le a_{j+1}$. If j is even, then $F_j \subseteq (ax \le a_0)$ implies

$$\begin{aligned} a_0 &\ge \frac{1}{2}(a_1 + a_2 + \dots + a_j) + a_{j+1} + y_{j+2}a_{j+2} + y_{j+3}a_{j+3} + \dots + y_n a_n \\ &\ge a_1 + a_3 + \dots + a_{j-1} + a_{j+1} + y_{j+2}a_{j+2} + y_{j+3}a_{j+3} + \dots + y_n a_n \end{aligned}$$

and

$$\begin{aligned} a_0 &\ge \frac{1}{2}(a_1 + a_2 + \dots + a_j) + a_{j+1} + y_{j+2}a_{j+2} + y_{j+3}a_{j+3} + \dots + y_n a_n \\ &\ge \frac{1}{2}a_1 + a_2 + a_4 + \dots + a_j + y_{j+2}a_{j+2} + y_{j+3}a_{j+3} + \dots + y_n a_n \\ &\ge a_2 + a_4 + \dots + a_j + y_{j+2}a_{j+2} + y_{j+3}a_{j+3} + \dots + y_n a_n \ . \end{aligned}$$

From the first sequence, we get that $u \in \{0,1\}^n$ with

$$u_i = \begin{cases} 1 & \text{if } i \le j+1 \text{ and } i \text{ odd} \\ 0 & \text{if } i \le j+1 \text{ and } i \text{ even} \\ y_i & \text{if } i \ge j+2 \end{cases}$$

is contained in $(ax \le a_0)$. The second sequence implies the same for $v \in \{0,1\}^n$ defined by

$$v_i = \begin{cases} 0 & \text{if } i \le j+1 \text{ and } i \text{ odd} \\ 1 & \text{if } i \le j+1 \text{ and } i \text{ even} \\ y_i & \text{if } i \ge j+2 \end{cases}$$

Consequently, both u and v are contained in $\operatorname{conv}\{x \in \{0,1\}^n \mid ax \le a_0\}$.

Similarly, if j is odd, then

$$\begin{aligned} a_0 &\geq \frac{1}{2}(a_1 + a_2 + \ldots + a_j) + a_{j+1} + y_{j+2}a_{j+2} + y_{j+3}a_{j+3} + \ldots + y_n a_n \\ &\geq a_1 + a_3 + \ldots + a_j + y_{j+2}a_{j+2} + y_{j+3}a_{j+3} + \ldots + y_n a_n \end{aligned}$$

and

$$\begin{aligned} a_0 &\geq \frac{1}{2}(a_1 + a_2 + \ldots + a_j) + a_{j+1} + y_{j+2}a_{j+2} + y_{j+3}a_{j+3} + \ldots + y_n a_n \\ &\geq a_2 + a_4 + \ldots + a_{j+1} + y_{j+2}a_{j+2} + y_{j+3}a_{j+3} + \ldots + y_n a_n \end{aligned}$$

imply again that u and v are points in $\operatorname{conv}\{x \in \{0,1\}^n \mid ax \leq a_0\}$. Therefore,

$$y = \frac{1}{2}(u + v) \in \operatorname{conv}\{x \in \{0,1\}^n \mid ax \leq a_0\} \ .$$

As a result, $y \in K(P)$. Now $F_1 \subseteq P$ implies that $F_n \subseteq K^{(n-1)}(P_n)$ and, hence, the knapsack-rank of P_n is at least n. Since the knapsack-rank cannot exceed n, it is exactly n. □

5.6 Complexity of the M-Closure

Considering the purpose of cutting plane methods, one of the most important questions regarding elementary closure operations is the question of whether one can efficiently optimize over a closure. Formulated differently, this is the question of the complexity of the separation problem associated with a family of cutting planes (see Definition 2.7). The following two problems are closely related:

Definition 5.24 *The validity problem for the elementary closure of a family of cuts is:*

> *Given a rational polyhedron $P \subseteq \mathbb{R}^n$ and a rational inequality $ax \leq a_0$, decide whether $ax \leq a_0$ is valid for the elementary closure of P for this family of cuts.*

Definition 5.25 *The membership problem for the elementary closure of a family of cuts is:*

> *Given a rational polyhedron $P \subseteq \mathbb{R}^n$ and a rational point $\bar{x} \in P$, decide whether $\bar{x}$ does not belong to the elementary closure of P for this family of cuts.*

In order to solve the membership problem and, in particular, to argue that a given point is not contained in the elementary closure, it is necessary to exhibit a violated inequality and, hence, to solve the validity problem. A characteristic of Gomory-Chvátal cutting planes is that their validity can be established easily, that is, one can verify in polynomial time that a given inequality is a Gomory-Chvátal cut for P. Consequently, both the membership and validity problem are in NP for the family of Gomory-Chvátal cuts. Eisenbrand (1999) showed that the membership problem for the Gomory-Chvátal closure is NP-complete. Hence, there is no polynomial algorithm for optimizing over the Gomory-Chvátal closure of a rational polyhedron, unless P=NP. In contrast, Bockmayr and Eisenbrand (1999) showed that for fixed dimension, the Gomory-Chvátal closure of a rational polyhedron P can be described by a polynomial number of inequalities. Moreover, in this case P' can be constructed in polynomial time. The same is true for the M-closure of a rational polytope with full-dimensional integer hull. This fact is a direct consequence of the facet characterization in Theorem 5.6.

Theorem 5.26 *If $P \subseteq [0,1]^n$ is a rational polytope with full-dimensional integer hull, the M-closure of $M(P)$ can be computed in polynomial time, when the dimension n is fixed.*

Proof. By Theorem 5.6, any facet of $M(P)$ is contained in $\mathcal{F}^n$. We have $|\mathcal{F}^n| \leq \binom{2^n}{n}$. For each $ax \leq a_0$ in $\mathcal{F}^n$, we can check in polynomial time whether it is a valid inequality for $M(P)$. For this, we first compute $z^* = \max\{ax \mid x \in P\}$ by solving a linear programming problem. Then we determine $z_I = \max\{ax \mid x \in \{0,1\}^n, ax \leq z^*\}$ by enumerating all 2^n cube vertices and compare the value with a_0. The inequality $ax \leq a_0$ is valid for $M(P)$ if and only if $z_I \leq a_0$. Since linear programming is solvable in polynomial time, and since the size of $\mathcal{F}^n$ and the size of $\{0,1\}^n$ are polynomially bounded in fixed dimension, the theorem follows. □

The situation is different when the M-closure is considered in general dimension. In order to certify that an inequality is an M-cut, it is necessary to solve a subset sum problem. Hence, the validity problem for the M-closure is already difficult.

Theorem 5.27 *The validity problem problem for the family of M-cuts is (weakly) NP-hard.*

Proof. We describe a reduction from the SUBSET SUM problem, which is NP-complete: Given a set of numbers $a_1, \dots, a_n \in \mathbb{Z}_+$ and a positive integer a_0, decide whether there is subset $I \subseteq \{1, \dots, n\}$ such that $\sum_{i \in I} a_i = a_0$. Consider an arbitrary instance of SUBSET SUM and let P be the polytope defined by $P := [0,1]^n \cap (ax \leq a_0)$. The inequality $ax \leq a_0 - 1$ is valid for $M(P)$ if and only if there is no point in $x \in P \cap \{0,1\}^n$ such that $ax = a_0$. In other words, $ax \leq a_0 - 1$ is valid for $M(P)$ if and only if the subset sum instance is a NO-instance. □

Chapter 6

Conclusions

In this dissertation, we have discussed several theoretical aspects of Gomory-Chvátal cutting planes and the associated Gomory-Chvátal closure of a polyhedron. While the general sentiment in the early 1990's was that generic cutting planes, and in particular Gomory-Chvátal cuts, were rather useless in practice, this perception began to change when these cutting planes were successfully implemented for the first time within a branch-and-cut framework (Balas, Ceria, Cornuéjols, and Natraj 1996). Even though Gomory-Chvátal cuts are weaker than various other cutting planes, their derivation is beautifully simple and, hence, they are very easy to compute. Besides these practical observations, the theory behind Gomory-Chvátal cuts is very elegant since they may be derived without any knowledge of the underlying structure of a problem. As a result, Gomory-Chvátal cutting planes have attracted the interest of many researchers in the integer programming community. In particular, polyhedral properties and complexity aspects of the Gomory-Chvátal closure have been the subject of investigation.

The work in this thesis contributes to a better understanding of the theory of Gomory-Chvátal cutting planes and, more precisely, to a better understanding of the geometry behind their derivation. One of our main contributions is to answer the question raised by Schrijver (1980) as to whether the Gomory-Chvátal closure of a non-rational polytope is itself a polytope. In Chapter 4, we give an affirmative answer to this question by proving that a finite number of Gomory-Chvátal cuts entirely describe the elementary closure of a non-rational polytope. In that sense, the chapter is conclusive. However, our proof technique for the key argument (Step 1) provides insights about the geometric behavior of Gomory-Chvátal cutting planes, as we identify intuitive geometric conditions for the validity of Gomory-Chvátal cuts. We believe that these observations might assist in approaching other aspects of the theory of Gomory-Chvátal cutting planes from a different perspective.

The most apparent open questions that arise from this work are in relation to Chapter 5, where we introduce a refined Gomory-Chvátal closure for polytopes in the unit cube, called the M-closure. The basic idea behind the definition of this new closure is very natural and also, from a computational point of view, reasonable: We strengthen Gomory-Chvátal cuts by shifting the associated hyperplanes towards the polytope until they intersect a 0/1 point. The shifting-operation corresponds to solving a knapsack problem, and such problem can be solved in pseudo-polynomial time. We exhibit examples of polytopes for which this strengthening results in a better approximation of the integer hull than the corresponding Gomory-Chvátal closure. Hence, the question as to whether the associated cutting plane procedure generates the integer hull within fewer iterations than the Gomory-Chvátal procedure is very interesting. The best known upper bound for the Chvátal rank of a polytope in the unit cube $[0,1]^n$ is $O(n^2 \log n)$ (see Eisenbrand and Schulz 2003). In the special case that there are no integer points in the polytope, a maximum of n iterations suffice. While we show that the latter bound also applies to the M-procedure, we were unable to prove a general better bound in the case where the integer hull of a polytope is non-empty. This rank question should be viewed in connection with Theorem 5.6, where we characterize the facet-defining inequalities of the M-closure of a polytope. In particular, we show that for a polytope $P \subseteq [0,1]^n$ with full-dimensional integer hull, every facet of its closure $M(P)$ is associated with a hyperplane that is spanned by n affinely independent 0/1 points. Hence, understanding how these special cuts are generated in the M-procedure based on the validity of other inequalities with this special property is crucial for solving the rank problem. The insights that we gained so far from our attempts towards proving a better upper bound than $O(n^2 \log n)$ on the M-rank of a polytope let us believe the following:

Conjecture 6.1 *For every polytope $P \subseteq [0,1]^n$, the M-rank is bounded by a function in $O(n^2)$.*

This conjecture is, in our eyes, very intriguing for future research. An answer to whether the conjecture is true would not only be very interesting in its own right, it would most certainly help to better understand the geometry of the unit cube.

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