Quantum Noise as an Entanglement Meter

Leonid Levitov

MIT and KITP UCSB

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Part I:
Quantum Noise as an Entanglement Meter
with Israel Klich (2008); arXiv: 0804.1377

Part II:
Coherent Particle Transfer in an On-Demand Single-Electron Source
with Jonathan Keeling and Andrei Shytov (2008)
arXiv: 0804.4281
Israel Klich
UCSB

Jonathan Keeling
Cambridge Univ.

Andrei Shytov
Utah
Density matrix

Pure state vs. mixed state

\[ |\Psi\rangle = \sum_i a_i |\psi_i\rangle. \]

\[ E(B) = \sum_{i,j} a_i^* a_j \langle i | B | j \rangle. \]

Density matrix

Landau 1927

\[ \rho = \sum_j p_j |\psi_j\rangle \langle \psi_j|, \]

\[ E(B) = Tr(\rho B). \]

Quantum-statistical entropy

von Neumann 1927

\[ S(\rho) = -Tr(\rho \ln \rho), \]
Entanglement Entropy

- Expresses complexity of a quantum state
- Describes correlations between two parts of a many-body system
- Useful in: field theory, black holes, quantum quenches, phase transitions, quantum information, numerical studies of strongly correlated systems

\[ S = - \text{Tr} \rho_A \log \rho_A \]
\[ \rho_A = \text{Tr}_B \rho, \quad V = A + B \]

Wilczek, Bekenstein, Vidal, Kitaev, Preskill, Cardy, Bravyi, Hastings, Verstraete, Klich, Fazio, Levin, Wen, Fradkin...
Can it be measured?

arXiv: 0804.1377

- Relate to the electron transport
- Quantum point contact (QPC) with transmission tunable in time
- Open and close “door” between reservoirs R, L, let particles from R & L mix
- Statistics of current fluctuations encode S!

$$S_L = -\text{Tr}_L (\rho_L(t) \log \rho_L(t))$$

$$\rho_L(t) = \text{Tr}_R (U(t) \rho(t = 0) U^+(t))$$
Current fluctuations, counting statistics

- Probability distribution of transmitted charge
- Recently measured up to 5\textsuperscript{th} moment in tunnel junctions, quantum dots and QPC (Reulet, Prober, Reznikov, Fujisawa, Ensslin)
- Well understood theoretically

\[
\chi(\lambda) = \sum_n P_n e^{i\lambda n} = \exp \left( \sum_m \frac{(i\lambda)^m C_m}{m!} \right)
\]
A universal relation between noise and entanglement entropy

Electron noise cumulants

\[ S = \sum_{m>0} \frac{\alpha_m}{m!} C_m, \quad \alpha_m = \begin{cases} (2\pi)^m |B_m|, & m \text{ even} \\ 0, & m \text{ odd} \end{cases}, \]

\[ B_m \text{ are Bernoulli numbers } (B_2 = \frac{1}{6}, \ B_4 = -\frac{1}{30}, \ B_6 = \frac{1}{42} \ldots), \]

\[ S = \frac{\pi^2}{3} C_2 + \frac{\pi^4}{15} C_4 + \frac{2\pi^6}{945} C_6 + \ldots \]

For free fermions Full Counting Statistics accounts for ALL correlations relevant for the entanglement entropy

True for arbitrary protocol of QPC driving
Example: abrupt on/off switching

- Counting statistics computed explicitly
- Only $C_2$ is nonzero
- Logarithmic charge fluctuations, logarithmic entropy
- Agrees with field-theoretic calculations
- Can use electric noise to measure central charge

Heuristically, number fluctuations in a time-dependent interval:

Space-time duality: use time window (door open/close) instead of space interval at a fixed time
### Possible Experimental Realization

- **Periodic switching:** particle fluctuations and entropy proportional to total time;
- **Fixed increment** $\Delta S$ per driving period;
- **DC shot noise** reproduces $\Delta S$:

$$C_2(N) \approx \frac{N}{\pi^2} \log \frac{\sin \pi \nu \omega}{\pi \nu \tau}, \quad \nu = 1/T$$

$$C_2 \propto N \quad \frac{dS}{dt} = \frac{1}{3} \nu \log \frac{\sin \pi \nu \omega}{\pi \nu \tau}$$

**For** $\nu=500$ Mhz, $T_{\text{noise}}=25$ mK
Step 1: Relate many-body and one-particle quantities

Projected density matrix (gaussian for thermal state):

\[ \rho_L \propto e^{-\tilde{H}_{ij} a_i^\dagger a_j} \]

\[ M_{ij} = \text{Tr}_L \rho_L a_i^\dagger a_j \]

\[ \tilde{H} = \log((I - M)M^{-1}) \]

\[ i, j \in L, \]

Find the entropy of an evolved state:

\[ S_L = -\text{Tr} \left( M \log M + (1 - M) \log(1 - M) \right) \]

\[ S_L = -\int_0^1 dz \mu(z) \left( z \log z + (1 - z) \log(1 - z) \right) \]

\[ \mu(z) = -\frac{1}{\pi} \text{Im} \text{Tr} \frac{1}{z - M + i0} = -\frac{1}{\pi} \partial_z \text{Im} \log \det(z - M + i0) \]
Step 2: Counting statistics yields same quantity $M$.

Functional determinant in an original form (LL, Lesovik '92)

\[ \chi(\lambda) = \det \left( 1 - n_\epsilon + n_\epsilon U^\dagger e^{i\lambda P_L} U e^{-i\lambda P_L} \right) \]

Scattering operator

Recently: Klich, Ivanov, Abanov, Nazarov, Vanevic, Belzig

\[
\chi(\lambda) = \det \left( (1 - M + Me^{i\lambda P_L})e^{-i\lambda n P_L} \right)
\]

\[
g(z) = \log \det(1 - M + Me^{i\lambda P_L})
\]

\[
z^{-1} = 1 - e^{i\lambda}
\]

\[
g(z) = \log \det(z - M) - \text{rank}(M) \log z
\]
The quantity M

- Matrix in the single-particle Hilbert space;
- Describes partition of the modes between A and B: either statistical or dynamical;
- Intrinsic to the Full Counting Statistics
- Provides spectral representation for the entropy
Step 3: Combine results 1 and 2

\[ \mu(z) = \frac{1}{\pi} \text{Im} \, \partial_z \, g(z) + \text{rank}(M) \delta(z) \]

\[ \log \chi = \sum \frac{(i\lambda)^m}{m!} C_m \]

\[ \mu(z) = \frac{1}{\pi} \sum_m \frac{C_m}{m!} \text{Im} \, \partial_z \left( i\pi + \log \frac{1 - z}{z} \right)^m + \text{rank}(M) \delta(z) \]

Entanglement entropy

Relation of \( \alpha_m \) to Bernoulli numbers:

\[ \mathcal{S} = \sum_{m=1}^{\infty} \frac{1}{m!} C_m \alpha_m \]

\[ \int_0^\infty \frac{u^{2m}}{\sinh^2 u} \, du = \pi^{2m} |B_{2m}| \]
The spectrum of $M$ for a non-unit QPC transmission

Dependence on the parameters of driving unchanged (up to a rescaling factor)

$S \sim \log \sin \pi \nu w$

$F = S(D)/S(1)$
Summary & Outlook

- Universal relation between entanglement entropy and noise
- A new interpretation of Full Counting Statistics
- Generalization to other entropies (Renyi, etc);
- Opens way to measure S by electric transport (by pulsing QPC through on/off cycle)

Realize in cold atoms: particle number statistics
Restricted vs. unrestricted entanglement
Interacting systems? Neutral modes?
A similar relation of entropy and noise (FCS) for Luttinger liquid is found
Part II
Coherent Particle Transfer in an On-Demand Single-Electron Source

with Jonathan Keeling and Andrei Shytov (2008)
arXiv: 0804.4281
Noiseless particle source

- Transfer a particle from a localized state to a continuum without creating other excitations
- Populate a one-particle state in a Fermi gas without perturbing the rest of the Fermi sea
- Minimally entangled states in electron systems: coherent, noiseless current pulses
- Extend notion of quantized electron states (quantum dots, turnstiles) to states that can travel at a high Fermi velocity
- Bosons? Luttinger liquids?
Eject a localized electron into a Fermi continuum in a noiseless fashion

Electron system:

Cold atoms:

Quantum Tweezers (one-atom optical trap in a quantum gas)
Experimental realization in a 1d QHE-edge electron system

Quantized current pulses in an On-Demand Coherent Single-Electron Source

Excitation content: particles and holes

The number of excitations: *unhappiness* = \( N_p + N_h \)
Minimize unhappiness?

Optimize driving so that

\[ N_{ex} = N_e + N_h = \text{min}, \quad \Delta N = N_e - N_h = 1 \]

Localized and delocalized particles indistinguishable: Excitation unavoidable? No.
Multilevel Landau-Zener problems, exact S-matrix

Our problem:
Continuous spectrum, arbitrary driving

Demkov-Osherov

Discrete states, linear driving
Time-dependent S-matrix

Gate voltage, tunnel coupling

\[
[i \partial_t - E(t)] \phi(t) = \sum_p \lambda(t) \psi_p(t),
\]

\[
[i \partial_t - \epsilon_p] \psi_p(t) = \lambda^*(t) \phi(t),
\]

Quasi 1D scattering channel representation:

In-state:

\[
\psi(t, x) = \sum_p e^{ipx} \psi_p(t) \quad \epsilon_p \rightarrow -i v_F \partial_x.
\]

\[
[i \partial_t - E(t)] \phi(t) = \lambda(t) \int dx \delta(x) \psi(t, x)
\]

\[
[i \partial_t + iv_F \partial_x] \psi(t, x) = \lambda^*(t) \delta(x) \phi(t)
\]

\[
\psi(t, x < 0) = \psi_0(t, x) = \frac{1}{\sqrt{2\pi}} e^{-i\epsilon'(t-x/v_F)}
\]

The S-matrix:

Out-state:

\[
U(\epsilon, \epsilon') = \int \frac{dt}{\sqrt{2\pi}} \psi(t, x > 0) \exp \left[ i\epsilon \left( t - \frac{x}{v_F} \right) \right]
\]
Find the S-matrix:

\[
\psi(t, x) = \psi_0 \left[ t - \frac{x}{v_F} \right] - \frac{i}{v_F} \lambda^* \left[ t - \frac{x}{v_F} \right] \phi \left[ t - \frac{x}{v_F} \right] \theta(x).
\]

\[
\begin{bmatrix}
i \partial_t - E(t) + i \frac{|\lambda(t)|^2}{2v_F}
\end{bmatrix}
\phi(t) = \lambda(t) \psi_0(t)
\]

\[
\phi(t) = -i \int_{-\infty}^{t} dt' \lambda(t') \psi_0(t') e^{X(t, t')}
\]

\[
i \partial_t X(t, t') = [E(t) - i \Gamma(t)/2] X(t, t')
\]

Resonance width:

\[
|\lambda(t)|^2 / \nu = \Gamma(t)
\]

ANSWER:

\[
U(\epsilon, \epsilon') = \delta(\epsilon - \epsilon') - \int_{-\infty}^{\infty} dt \int_{-\infty}^{t} dt' \frac{\lambda(t) \lambda(t')}{2\pi v_F} e^{A(t, t')}
\]

\[
A(t, t') = i(\epsilon t - \epsilon' t') - \int_{t'}^{t} d\tau \left[ \frac{\Gamma(\tau)}{2} + i E(\tau) \right]
\]
Number of excitations

Energy representation:

\[ N^+ = \langle \Omega | U^\dagger \sum_{\epsilon > \epsilon_F} a_\epsilon^\dagger a_\epsilon U | \Omega \rangle = \int_{\epsilon_F}^{\infty} d\epsilon \int_{-\infty}^{\epsilon_F} d\epsilon' | U(\epsilon, \epsilon')|^2 \]

Time representation:

\[ N^+ = -\left( \frac{\Gamma}{2\pi} \right)^2 \int_{-\infty}^{\infty} dt \int_{-\infty}^{t} dt' \int_{-\infty}^{\infty} ds \int_{-\infty}^{s} ds' \exp \left[ -\frac{\Gamma}{2} (t - t' + s - s') - i \int_{t'}^{t} E(\tau) d\tau + i \int_{s'}^{s} E(\tau) d\tau \right] \]

\[ \frac{(t - s + i0)(t' - s' + i0)}{(t - s + i0)(t' - s' + i0)} \]

Excitation number depends on the protocol, E(t)
Optimal driving?
Linear driving minimizes unhappiness

Slow or fast rapidity, degeneracy in c

Resulting state depends on c value

Relevant energy window: \(|\varepsilon - \varepsilon_F|\) of order \(\Gamma\)
S-matrix for linear driving

\[ A(T, \tau) = i(\epsilon + \epsilon') \frac{\tau}{2} - i(\epsilon' - \epsilon) T - \frac{\Gamma}{2} \tau - icT \tau. \]

\[ t = T + \tau/2, \quad t' = T - \tau/2, \text{ with } \tau > 0 \]

S-matrix: rank-one particle/hole block

\[ U(\epsilon \neq \epsilon') = \theta(\epsilon - \epsilon') \frac{\Gamma}{c} e^{-\frac{\Gamma}{2c} (\epsilon - \epsilon')} + \frac{i}{2c} (\epsilon^2 - \epsilon'^2). \]

\[ N^+ = 1, \quad N^- = 0, \quad N^+ - N^- = 1 \]

No e/h pairs: \( U_{ab} U_{a'b'} - U_{ab'} U_{a'b} = 0 \)
Current pulse profile at different rapidities

\[ \psi(t, x) = \sqrt{\frac{\Gamma}{c}} \int_{0}^{\infty} \frac{d\varepsilon}{\sqrt{2\pi}} \exp \left[ -i \varepsilon \left( \frac{t - x}{v_F} \right) - \frac{\Gamma \varepsilon}{2c} + i \frac{\varepsilon^2}{2c} \right] \]

- **High c:** exponential profile
- **One-electron pulse with fringes on the trailing side**
- **Low c:** Lorentzian profile
Energy excitation and e/h pair production suppressed by Fermi statistics

Pauli principle helps to eliminate entanglement
Use noise to measure unhappiness

- Send current pulses on a QPC (beamsplitter): The partition noise generated at QPC is a direct measure of the excitation number.
- Use a periodic train of pulses, vary frequency, protocol, duty cycle, etc, to demonstrate noise minimum.
- At finite temperature must have $\hbar \nu > kT$: e.g. $T = 10$ mK, $\nu > 200$ MHz.
More examples

- Harmonic driving, \( E(t) = E_0 + \cos \Omega t \), simulates repeated linear driving;
- Linear driving + classical noise:
  \[
  E(t) = ct + \delta V(t),
  \]
  \[
  <\delta V(t)\delta V(t')> = \gamma^2 \delta(t-t')
  \]

Total number of excitations:

\( N_{\text{ex}} = 1 \) for fast driving;

\[
N_{\text{ex}} \approx \frac{2\gamma \gamma^2}{\pi c} \ln \frac{\omega_0}{\gamma_*}
\]

for slow driving (multiple crossings of the Fermi level);

Crossover at \( c \sim \gamma \gamma^2 \)
A more intuitive picture at slow driving: quasistationary time-dependent scattering phase

\[ \theta(t) = \arctan\left( \frac{\varepsilon - E(t)}{\Gamma} \right) \]

Translates into an effective time-dependent ac voltage:

\[ V(t) = \left( \frac{h}{e} \right) \frac{d\theta}{dt} \]

Noiseless excitation realized for Lorentzian pulses of quantized area (PRL 97, 116403 (2006))
Clean excitation by a voltage pulse

- Particle excited above $E_F$;
- Other particles filling the void at $E < E_F$ (near $+k_F$);
- Undisturbed Fermi sea (no mess left behind);
- Counter-propagating hole (similar, near $-k_F$).
Minimal noise requirement

\[ N_{ex} = N_e + N_h \rightarrow \min, \quad \Delta q/e = N_e - N_h = n = \text{const} \]

- An interesting variational problem, solved by pulses of integer area \(2\pi n\):

\[
V(t) = \frac{\hbar}{e} \sum_{i=1 \ldots n} \frac{2\tau_i}{(t - t_i)^2 + \tau_i^2} \quad (\tau_i > 0)
\]

Lorentzian pulses (overlapping or non-overlapping): \(N_h = 0\) or \(N_e = 0\)

Degeneracy: \(N_{ex} = n\), the same for all \(t_i, \tau_i\)
When does a unitary evolution excites at most one particle?

Evolve a Fermi sea, \( n = \sum_{E_k < E_F} |k\rangle\langle k| \):

\[
n \to UnU^{-1}, \quad U_{+-} = (1 - n)Un, \quad U_{-+} = nU(1 - n)
\]

Criterion: IF and ONLY IF \( U_{-+} = 0, U_{+-} = c|\phi_+\rangle\langle \phi_-| \) a rank one matrix.

Proof: \( \langle k'|U|k\rangle = \langle k'|\phi_+\rangle\langle \phi_-|k\rangle \) for \( E_k < E_F, E_{k'} > E_F \);

\[
U_{a\rightarrow a'}U_{b\rightarrow b'} - U_{a\rightarrow b'}U_{b\rightarrow a'} = 0, \quad \text{at most one particle excited.}
\]

Transition amplitude for Lorentzian pulses \( \psi(t, x) = \psi(0, x + vt)e^{i\phi(t)} \):

\[
e^{i\phi(t)} = \frac{t + i\xi_k^*}{t - i\xi_k}, \quad \xi_k = \tau_k - it_k
\]

Fourier transform: \( \int e^{i\phi(t) + i\omega t}dt = \delta(\omega) + \sqrt{2\tau}e^{-\xi\omega}\theta(\omega), \omega = E_{k'} - E_k \);

Criterion fulfilled due to multiplicativity of exp!
FEATURES:

- A many body excitation which conspires to behave like a single particle;
- Direct product of $e$ and $h$;
- Energy distribution width $\hbar/\tau$ — inverse pulse width;
- Generalized to many pulses of equal sign. "Laughlin" algebra:

$$\prod_{k=1}^{n} e^{i\phi_k(t)} |0\rangle = \prod_{k<k'} \frac{\xi_k + \xi_{k'}^*}{\xi_k - \xi_{k'}^*} A_n^\dagger A_{n-1}^\dagger \cdots A_1^\dagger |0\rangle, \quad A_k^\dagger = \sum_{\epsilon > E_F} e^{-\xi_k \epsilon} a_k^\dagger$$

- Pulses of opposite sign: entangled e-h pairs and an undisturbed Fermi sea;

$$e^{-i\phi_1(t)} e^{i\phi_2(t)} |0\rangle = \frac{\xi_k - \xi_{k'}^*}{\xi_k + \xi_{k'}^*} A_1^\dagger B_2^\dagger |0\rangle + \frac{2\sqrt{T_k T_{k'}}}{\xi_k + \xi_{k'}^*} |0\rangle$$

- Generalized for chiral Luttinger liquid (QHE edge state):

$e \to e_* = e/m, \int V dt = \hbar/e_*$ — fractional charge pulses.
Summary

- Many-body states that conspire to behave like one-particle states
- Release/trap a particle in/from a Fermi sea in a clean, noiseless way
- Single-particle source can be realized using quantum dots: a train of quantized pulses of high frequency
- Can employ particle dynamics with high Fermi velocity $10^8$ cm/s to transmit quantized states in solids