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A necessary condition for decoupling multivariable systems†

MICHAEL E. WARREN‡ and SANJOY K. MITTER§

The decoupling of linear, time-invariant, multivariable systems into single-input, multiple-output subsystems is considered and a theorem by Falb and Wolovich completely solving a special case of this problem is extended to provide a strong necessary condition for decoupling. Unlike other results establishing necessary and sufficient conditions for this problem, the test developed herein is easily applied and does not require extensive constructions. Finally it is shown that this result extends easily to the more general case of decoupling a linear system into multivariable subsystems.

1. Introduction

The problem of decoupling linear, time-invariant, multivariable systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) \tag{1}$$

into smaller order subsystems by memoryless feedback control laws

$$u(t) = Fx(t) + Gv(t) \tag{2}$$

where $x(t) \in R^n$, $u(t)$, $v(t) \in R^m$, and $y(t) \in R^q$ with A, B, C, F, and G appropriately dimensioned real matrices, has been considered in the control literature for nearly a decade. Morgan (1964), generally credited for putting the problem in a state variable framework, developed a sufficient condition for decoupling when $m = q$ and the desired subsystems are all single input, single output. Three years later Falb and Wolovich (1967) completely solved this question, showing that decoupling was possible if and only if an easily constructed matrix, dependent upon A, B, and C, was non-singular. For ease of notation we will refer to this issue as Morgan's Problem.

Beginning in early 1970, Wonham and Morse (1970, 1971) promoted a new and more general theory of decoupling. Using geometric methods they were able to formulate the problem of decoupling a system (1) into other than simply single-input, single-output subsystems, and obtained complete solutions for several special cases, including Morgan's Problem.

Although the Wonham and Morse results are more powerful than, and subsume all earlier results, their geometric conditions require involved subspace constructions and are not as easily applied to a given linear system as those of Falb and Wolovich. It would therefore be useful if the method of Falb and Wolovich could be extended to a more general class of decoupling problems, even those already solvable by geometric methods. By considering a theorem of Morse and Wonham regarding the decoupling of linear systems

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‡ Department of Electrical Engineering, University of Florida.

§ Electronic Systems Laboratory, M.I.T.

into single-input, multiple-output subsystems, we are able to derive a strong necessary condition in the form of the Falb and Wolovich result for this more general problem. In addition, by example we show that no further information pertaining to decoupling may be extracted by such a formulation. Finally, we show that this necessary condition easily extends to the case of decoupling a linear system into general multivariable subsystems.

Our attempt is not to duplicate, albeit in a different manner, previously established results. Wonham and Morse (1970), Sato and LoPresti (1971) and Silverman and Payne (1971) have all examined and solved this problem. However, the conditions of Sato and LoPresti, as well as those of Silverman and Payne, involve complicated algorithms which are not easily applied. By exploiting a geometric result we obtain a strong, yet readily implemented necessary condition for decoupling, and hence achieve a significant reduction in complexity for an important partial solution to this problem.

2. Problem formulation

For any positive integer k, let $\mathbf{k}$ denote the set of integers $\{1, 2, \ldots, k\}$. Suppose that for a linear system of the form (1), the output $y(t)$ consists of k subvectors, $y_i(t) = C_i x(t)$, where C_i is $q_i \times n$ and $C = (C_1' ; \ldots ; C_k')'$. (The superscript prime denotes matrix transpose.) Such a system may be decoupled if there exists a feedback control law

$$u(t) = Fx(t) + \sum_{i \in \mathbf{k}} G_i v_i(t) \tag{3}$$

such that input $v_i(t)$ controls suboutput $y_i(t)$ and affects no other suboutputs $y_j(t)$, $j \neq i$, $i \in \mathbf{k}$.

For Morgan's Problem we have $q = k = m$, whence the inputs $v_i(t)$ and outputs $y_i(t)$ are all scalars, and the matrices G_i represent the columns of G from (2). Morgan's Problem is then solved if we can find an (F, G) pair such that $H(\lambda) = C(\lambda I - A - BF)^{-1} BG$ is diagonal and non-singular (Gilbert 1969).

The method proposed by Falb and Wolovich is relatively straightforward. For each $i \in \mathbf{m}$ define the non-negative integer d_i and row vector D_i as follows :

$$d_i = \min \{ j / C_i A^j B \neq 0, \quad j = 0, 1, \ldots, n-1 \}$$

$$d_i = n - 1 \quad \text{if } C_i A^j B = 0 \quad \text{for all } j \geqslant 0 \tag{4}$$

$$D_i = C_i A^{d_i} B \tag{5}$$

The system (1), denoted by the matrix triple (A, B, C), may be decoupled if and only if the vectors D_i', $i \in \mathbf{m}$ are linearly independent. If this is true, then letting D be the $m \times m$ matrix whose ith row is D_i, i.e. $D = [D_1' ; \ldots ; D_m']'$ and A^* the $m \times n$ matrix whose ith row is given by $C_i A^{d_i+1}$ for $i \in \mathbf{m}$, the control law

$$u(t) = -D^{-1} A^* x(t) + D^{-1} v(t)$$

will decouple (A, B, C) into a particularly simple form. (For proofs see Falb and Wolovich (1967) or Gilbert (1969).)

The geometric methods of Wonham and Morse depend heavily on the concept of a controllability subspace (c.s.). A subspace $\mathscr{R}$ is said to be a c.s. if for some feedback map F

$$\mathscr{R} = \sum_{i \in \mathbf{n}} (A + BF)^{i-1}(\mathscr{B} \cap \mathscr{R})$$

where $\mathscr{B}$ denotes the image of the map B (Wonham and Morse 1970). It is immediate that a c.s. is invariant under the action of $A + BF$ and is a completely reachable subspace.

Among the decoupling problems for which Wonham and Morse have developed complete answers is the case where rank $B = k$, e.g. the desired subsystems are all single input but possibly multiple output. If we let $\mathscr{R}_i$ denote the maximal c.s. contained in the common kernel of the suboutput maps $C_1, \ldots, C_{i-1}, C_{i+1}, \ldots, C_k$

$$\mathscr{R}_i \subset \bigcap_{\substack{j \neq i \\ j \in \mathbf{k}}} \mathscr{N}_j, \quad i \in \mathbf{k}$$

where $\mathscr{N}_j = \operatorname{Ker} C_j$, then it may be shown (Morse and Wonham 1971, Theorem 8) that (A, B, C) is decoupleable if and only if

$$\mathscr{B} = \sum_{i \in \mathbf{k}} (\mathscr{B} \cap \mathscr{R}_i) \tag{6}$$

Although (6) represents a compact and complete test for the decoupleability of a system (A, B, C) with suitably partioned output, the calculation of the maximal c.s. $\mathscr{R}_i$, $i \in \mathbf{k}$ requires significant computational effort. In addition the computations may be highly sensitive to parameter variations in the elements of A, B or C. For these reasons we attempt to develop extensions of the basic Falb–Wolovich results in the succeeding sections of this paper.

3. Decoupling with an excess of inputs

We first consider a relatively easy and direct generalization of Morgan's Problem to the case where rank $B = m \geqslant q$. Since the suboutputs are still assumed scalar, it follows that several inputs may control a given output, or some inputs may affect more than one output. In the latter case, by inactivating such inputs, it may still be possible to effect complete noninteraction. Alternatively, one may consider feedback laws of the form (2) with G possibly singular as admissible candidates for decoupling.

We define the integers d_i and row vectors D_i as before by (4) and (5). Now constructing the $q \times m$ matrix D whose ith row is given by D_i for $i \in \mathbf{q}$ we have the following result.

Proposition 1

The system (A, B, C) may be decoupled if and only if D contains a $q \times q$ non-singular submatrix.

The proof is straightforward and may be found in Warren (1974 a).

We note that if the system is decoupleable, a feedback law which leaves the decoupled system in particularly simple form may be synthesized from the matrix D. Indeed let R denote the non-singular $q \times q$ submatrix of D

consisting of columns $i_1, \ldots, i_q$, and define A^* as before. If we let s_j represent the jth row of $-R^{-1}A^*$, and f_p the pth row of F, then choose $f_{i_j}=s_j$ for $j\in\mathbf{q}$, and zero otherwise. Similarly if we let r_j represent the jth row of R^{-1} and g_p the pth row of G, then choose $g_{i_j}=(r_j\ 0)$ for $j\in\mathbf{q}$, zero otherwise. Note that if $D=[R : Q]$, then F and G are simply

$$\left(\begin{array}{c} -R^{-1}A^* \\ \hline 0 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{c|c} R^{-1} & 0 \\ \hline 0 & 0 \end{array}\right)$$

respectively.

4. Decoupling into single-input, multiple-output subsystems

We now move on to our main result, establishing an extension of the Falb and Wolovich theorem to the case of decoupling into single-input, multiple-output (SIMO) subsystems. To simplify the statement of our condition, it is helpful to first settle upon precise notation.

Our attention is focused upon a controllable linear system of the form (1) with output $y(t)$ consisting of m subvectors $y_i(t)=C_ix(t)$, where C_i is a $q_i\times n$ matrix

$$C_i=\begin{bmatrix} C_{i1} \\ \vdots \\ C_{iq_i} \end{bmatrix}, \quad i\in\mathbf{m}$$

with C_{ij} a linear form (row vector) on R^n. (Note that we implicitly assume B is of full rank m.) For each row C_{js_j} of C, $s_j\in\mathbf{q}_j$, $j\in\mathbf{m}$, we may define the feedback invariant d_{js_j} as in (4), and then construct the augmented matrix D as follows.

For each $i\in\mathbf{m}$, define the $q_i\times m$ matrix D_i given by

$$D_i=\begin{bmatrix} C_{i1}A^{d_{i1}}B \\ \vdots \\ C_{iq_i}A^{d_{iq_i}}B \end{bmatrix}$$

Then we may synthesize D, a $q\times m$ matrix wholly dependent upon A, B and C from the D_i's,

$$D=\begin{bmatrix} D_1 \\ \vdots \\ D_m \end{bmatrix}$$

Further, for each $i\in\mathbf{m}$, construct the $(q-q_i)\times m$ submatrix D_i^* of D by

$$D_i^*=\begin{bmatrix} D_1 \\ \vdots \\ D_{i-1} \\ \\ D_{i+1} \\ \vdots \\ D_m \end{bmatrix}$$

and finally for each set of m integers $(s_1, \ldots, s_m)$ with $s_j \in \mathbf{q}_j$, $j \in \mathbf{m}$, we define the $m \times m$ submatrix $D_{(s_1, \ldots, s_m)}$ of D:

$$D_{(s_1, \ldots, s_m)} = \begin{bmatrix} C_{1s_1} A^{d_{1s_1}} B \\ \vdots \\ C_{ms_m} A^{d_{ms_m}} B \end{bmatrix}$$

With this notation behind us, we may state our main result.

Theorem 1

A linear system may be decoupled into SIMO subsystems only if:

(i) for every set $(s_1, \ldots, s_m)$ with $s_j \in \mathbf{q}_j$, $j \in \mathbf{m}$ the submatrix $D_{(s_1, \ldots, s_m)}$ has rank m;

(ii) for every $i \in \mathbf{m}$, the submatrix $D_i{}^*$ has rank $m-1$.

Proof

The proof of this theorem is rather lengthy and is relegated to the Appendix.

Theorem 1 provides us with a readily implementable yet strong test which must be satisfied before a linear system may be decoupled into SIMO subsystems. Condition (i) says that every m-imput, m-output subsystem of the original system, consistent with the desired partition of the outputs (i.e. one output per output block) must be decoupleable into m single-input, single-output subsystems itself. Indeed the transfer function of a system decoupled into SIMO subsystems would necessarily be of the form

$$H(\lambda\,;\ F, G) = \text{block diagonal } [h_1(\lambda\,;\ F, G), \ldots, h_m(\lambda\,;\ F, G)]$$

with $h_i(\lambda\,;\ F, G)$ $q_i \times 1$, for $i \in \mathbf{m}$. Control of the outputs requires that for each $i \in \mathbf{m}$, every component of $h_i(\lambda, F, G)$ is non-zero. Thus for every m element subset of the q outputs, $(y_{1s_1}, \ldots, y_{ms_m})$, where $s_i \in \mathbf{q}_i$, $i \in \mathbf{m}$, the corresponding rows of $H(\lambda\,;\ F, G)$ form an $m \times m$ diagonal non-singular matrix. As the rows of D may be determined from the rows of H, this last fact implies that the submatrix $D_{(s_1, \ldots, s_m)}$ has rank m.

It is easily shown that the rows of the matrix D are feedback invariants (see Gilbert (1967) for a proof). Thus if condition (ii) of the theorem is violated, i.e. if $D_i{}^*$ has rank m for any $i \in \mathbf{m}$ it follows that all m inputs affect the $m-1$ output subvectors $(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_m)$ in a non-trivial manner. Since this interaction cannot be eliminated by feedback, the system may not be decoupled.

From the results on the generic solvability of decoupling problems by Fabian and Wonham (1973) and Warren (1974 b), we recognize that linear systems are not generically decoupleable into SIMO subsystems, unless $q_i = 1$ for all $i \in \mathbf{m}$. Thus, given a system (A, B, C) of the form considered in this section, the requirements of the theorem are less likely to hold as the number of outputs, and hence the number of components per suboutput vector, q_i, $i \in \mathbf{m}$ increases.

Unlike the geometric condition (6) on Wonham and Morse, Theorem 1 states only a necessary condition for decoupling. Indeed there is not sufficient information inherent in the augmented D matrix to develop a complete solution based only upon this matrix. For Morgan's Problem it is easy to show that D non-singular guarantees the existence of a feedback law (F, G) such that the effects of all inputs other than input i could be localized to $\operatorname{Ker} C_i$, for $i \in \mathbf{m}$. This could be accomplished as the subspaces $\operatorname{Ker} C_i$ were all of dimension $n-1$. In the more general problem at hand, since dimension $\operatorname{Ker} C_i$ may be less than $n-1$, the number of inputs may not be sufficient to afford the freedom required to simultaneously localize the effect of each input.

At this point let us examine several examples to help clarify the results presented here. Consider the system represented by the matrix triple

$$A=\begin{pmatrix}1 & 0 & 0\\ 1 & 1 & 1\\ 0 & 0 & 1\end{pmatrix},\quad B=\begin{pmatrix}1 & 0\\ 0 & 0\\ 0 & 1\end{pmatrix},\quad C_1=\begin{pmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{pmatrix}',\quad C_2=\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}'$$

We may readily construct the augmented D matrix as $d_{11}=d_{21}=0$ and $d_{12}=1$,

$$D=\begin{pmatrix}1 & 0\\ 1 & 1\\ \hline 0 & 1\end{pmatrix}$$

However, as the submatrix $D_2{}^*=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$ has rank 2, this system cannot be decoupled. Indeed, using the constructions of Wonham and Morse, we find $\mathscr{R}_2$, the maximal c.s. contained within $\operatorname{Ker} C_1$ is the zero subspace.

As a second example let us now consider the system represented by

$$A=\begin{bmatrix}a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}\end{bmatrix},\quad B=\begin{bmatrix}0 & 0\\ 1 & 0\\ 0 & 0\\ 0 & 1\end{bmatrix},\quad C_1=\begin{bmatrix}1\\ 1\\ 0\\ 0\end{bmatrix}',\quad C_2=\begin{bmatrix}0 & 0\\ 0 & 0\\ 1 & 0\\ 1 & 1\end{bmatrix}'$$

where the a_{ij}'s are temporarily unspecified. We may readily construct the augmented D matrix as it is independent of A (all the d_{ij}'s are zero),

$$D=\begin{bmatrix}1 & 0\\ \hline 0 & 1\\ 0 & 1\end{bmatrix}$$

and note that both requirements of Theorem 1 are satisfied. As we might expect, the existence of a decoupling feedback law will hinge on the values given to the elements of A.

From the form of D, it follows if the control law (F, G) decouples (A, B, C) G must be diagonal. Letting $\mathscr{B}_i$ denote the subspace spanned by the ith column of B, $i \in \mathbf{2}$, we need only find a feedback map F such that

$$\sum_{k \in \mathbf{n}} (A+BF)^{k-1}\mathscr{B}_i \subset \operatorname{Ker} C_j, \quad j \neq i, \quad i, j \in \mathbf{2}$$

Assume a most general feedback map

$$F = \begin{bmatrix} f_1 & f_2 & f_3 & f_4 \\ f_5 & f_6 & f_7 & f_8 \end{bmatrix}$$

Then it is immediately established that no F exists such that

$$\sum_{k \in \mathbf{n}} (A+BF)^{k-1}\mathscr{B}_1 \subset \operatorname{Ker} C_2$$

for all possible A as

$$C_2(A+BF)B_1 = \begin{bmatrix} a_{32}+a_{42}+f_6 \\ a_{42}+f_6 \end{bmatrix}$$

and hence in particular this system cannot be decoupled if $a_{32} \neq 0$.

Now fix $a_{12}=a_{14}=a_{21}=a_{24}=a_{34}=a_{44}=1$, and set all the other elements of A to zero. It follows that (A, B) is controllable. Now set $f_1=f_2$, $f_4=-2$, $f_3=f_5=f_6=0$. Then

$$A+BF = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1+f_1 & f_1 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & f_7 & 1+f_8 \end{bmatrix}$$

and we have

$$\sum_{k \in \mathbf{n}} (A+BF)^{k-1}\mathscr{B}_1 = \operatorname{Span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ f_1 \\ 0 \\ 0 \end{bmatrix}\right\} \subset \operatorname{Ker} C_2$$

$$\sum_{k \in \mathbf{n}} (A+BF)^{k-1}\mathscr{B}_2 = \operatorname{Span}\left\{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1+f_8 \end{bmatrix}, \begin{bmatrix} f_8 \\ -f_8 \\ 1+f_8 \\ f_7+(1+f_8) \end{bmatrix}\right\} \subset \operatorname{Ker} C_1$$

Clearly then the system $(A+BF, B, C)$ is decoupled.

As a final example consider the fifth-order, three-input, four-output linear system represented by the matrix triple (A, B, C):

$$A=\begin{bmatrix} 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 3 & 0 & 1 \\ 1 & 2 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 & 1 \\ -3 & 0 & 1 & 0 & 2 \end{bmatrix}, \quad B=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 3 \end{bmatrix}, \quad C_1=\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}'$$

$$C_2=\begin{bmatrix} 0 \\ 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}', \quad C_3=\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}'$$

It may be verified that this system is controllable, and that the augmented D matrix is

$$D=\left[\begin{array}{ccc} 1 & -1 & 0 \\ \hline 0 & 1 & 2 \\ \hline 1 & 0 & 2 \\ 1 & 1 & 4 \end{array}\right]$$

with $d_{11}=d_{21}=d_{32}=0$, $d_{31}=1$. We note that the submatrices D_i^*, $i \in \mathbf{3}$ are all of rank 2, hence condition (ii) of Theorem 1 is satisfied. However, $D_{(1,1,1)}$ is singular as is $D_{(1,1,2)}$ and thus this system may not be decoupled.

5. A more general decoupling problem

We are now in a position to combine the results of the previous two sections and extend them to the problem of decoupling a linear system into multiple-input, multiple-output (MIMO) subsystems. Specifically we assume a system of the form (1) with k output subvectors, $y_i(t)=C_i x(t)$, $i \in \mathbf{k}$, but now we allow the number of inputs, m to exceed k, $m \geqslant k$. For this more general problem to be solvable, conditions similar to those of Theorem 1 must hold.

Indeed, for such a system we may readily construct an augmented $q \times m$ matrix D with submatrices D_i^* for $i \in \mathbf{m}$, and $D_{(s_1, \ldots, s_k)}$, with $s_i \in \mathbf{q}_i$, $i \in \mathbf{k}$ defined as in the preceding section. Then if the system is decoupleable, it follows that the feedback invariant portion of the responses of any set of k suboutputs $(y_{1s_1}, \ldots, y_{ks_k})$ where $s_i \in \mathbf{q}_i$, $i \in \mathbf{k}$ must be determined by k independent inputs. In other words, the submatrix $D_{(s_1, \ldots, s_k)}$ must be of rank k. (This is most easily seen if we consider a discrete system formulation

$$x(s+1) = Ax(s) + Bu(s), \quad y(s) = Cx(s)$$

in which case the first non-zero response at suboutput y_{ij} due to input $u(s)$ is given by $D_{ij}u(s)$.)

In the light of Proposition 1 this is quite reasonable. Condition (i) of Theorem 1 states that every problem of Morgan's type embedded in the original problem must be solvable. For the case $m \geqslant k$, this becomes every m-input, k-output subproblem consistently embedded must be solvable. By Proposition 1 it follows that every submatrix $D_{(s_1, \ldots, s_k)}$ must have rank k.

Further, if for any $i \in \mathbf{k}$, the matrix D_i^* has rank m, then all m inputs affect the $k-1$ suboutput vectors $(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_k)$ in a manner which cannot be remedied by feedback. Of course if this happens, the system in equation cannot be decoupled.

Corollary 1

A linear system with $m \geqslant k$, $q_i \geqslant 1$, $i \in \mathbf{k}$ may be decoupled into k MIMO subsystems only if :

(i) for every set $(s_1, \ldots, s_k)$ with $s_i \in \mathbf{q}_i$, $i \in \mathbf{k}$, the submatrix $D_{(s_1, \ldots, s_k)}$ has rank k ;

(ii) for every $i \in \mathbf{k}$, the submatrix D_i^* has rank not exceeding $m-1$.

The proof of Corollary 1 follows directly from Theorem 1 and the preceding discussion.

As an illustration of the corollary, let us consider an example provided by Morse and Wonham (1971). The system matrices are given by

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}', \quad C_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}'$$

The augmented D matrix is seen immediately to be

$$D = \begin{bmatrix} 0 & 1 & 0 \\ \hline 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

and hence satisfies the conditions of the corollary. This system may indeed be decoupled, although the geometric constructions of maximal c.s. are not of themselves useful in reaching this conclusion.

6. Conclusions

The theorem of Falb and Wolovich concerning the solution of Morgan's Problem is extended to provide a strong necessary condition for the decoupling of a linear system into single-input, multiple-output subsystems. Our starting point is a theorem by Wonham and Morse which completely solves this problem, but which nevertheless is not easily applied to arbitrary systems. The derived condition, however is easily tested, eliminating this computational difficulty.

It is shown by example that a condition such as the one derived, formulated along the lines of the original Falb–Wolovich result, cannot provide a sufficiency result for this type of problem. However, the necessary condition easily generalizes to the case of decoupling into arbitrary multivariable subsystems.

Appendix

Proof of Theorem 1

The proof of Theorem 1 is accomplished via four lemmas which yield a series of necessary conditions culminating in the desired result. Lemmas A 2, A 3, and A 4, as given in this Appendix establish both necessary and sufficient conditions and are thus stronger than required to alone prove the theorem ; indeed only the necessity of these results is essential for our stated goal. Nevertheless, the sufficiency arguments in these three lemmas provide additional insight into the structure of the problem and are included primarily for this reason.

Before proceeding to the development of the proof, it is necessary to establish some notation which will ultimately simplify our task. We are concerned with a system of the form (1) where the output consists of k subvectors, $k=m$, the number of independent columns in B. Each output subvector is given by $y_i(t)=C_i x(t)$ with $C_i=[C_{i1}' \,;\, \dots \,;\, C_{iq_i}]'$.

Define $\mathscr{N}_{is}=\operatorname{Ker} C_{is}$ for $s\in\mathbf{q}_i$, and $i\in\mathbf{m}$, with $\mathscr{N}_i=\operatorname{Ker} C_i$, for $i\in\mathbf{m}$. Then $\mathscr{N}_i=\bigcap_{s\in\mathbf{q}_i}\mathscr{N}_{is}$. We let $\mathscr{K}_i=\bigcap_{\substack{j\neq i\\ j\in\mathbf{m}}}\mathscr{N}_j$, and denote the maximal (A, B) invariant subspace (i.s.) contained in $\mathscr{K}_i$ by $\mathscr{V}_i$, for $i\in m$. (A subspace $\mathscr{V}$ is said to be (A, B) invariant if $A\mathscr{V}\subset\mathscr{B}+\mathscr{V}$. This condition is equivalent to the existence of a map F such that $(A+BF)\mathscr{V}\subset\mathscr{V}$.) Finally, let $\mathscr{V}_{is}$ be the maximal (A, B) i.s. contained in $\mathscr{N}_{is}$ for $s\in\mathbf{q}_i$, and $i\in\mathbf{m}$. Then since $\mathscr{N}_{is}$ is always of co-dimension 1, it follows from Morse and Wonham (1971) that

$$\mathscr{V}_{is}^{\perp}=\mathscr{S}_{is}+z_{is} \quad \text{for } s\in\mathbf{q}_i, \text{ and } i\in\mathbf{m}$$

$$\mathscr{S}_{is}=\operatorname{Span}\{C_{is}', \dots, (A')^{d_{is}-1}C_{is}'\}$$

$$z_{is}=\operatorname{Span}\{(A')^{d_{is}}C_{is}'\}$$

where d_{is} is the least non-negative integer such that $C_{is}A^{d_{is}}B\neq 0$ for $s\in\mathbf{q}_i$, and $i\in\mathbf{m}$, and $\mathscr{S}_{is}\triangleq\{0\}$ if $d_{is}=0$. Letting $\mathscr{N}=\operatorname{Ker} B'$, we note that $\mathscr{S}_{is}\subset\mathscr{N}$ and $z_{is}\cap\mathscr{N}=0$ for $s\in\mathbf{q}_i$, and $i\in\mathbf{m}$.

Let us now prove the first step.

Lemma A 1

The geometric condition $\mathscr{B}=\sum_{i\in\mathbf{m}}(\mathscr{B}\cap\mathscr{R}_i)$, where $\mathscr{R}_i$ is the maximal c.s. contained in $\mathscr{K}_i$, $i\in\mathbf{m}$ implies

$$\mathscr{N}=\bigcap_{i\in\mathbf{m}}\Big(\mathscr{N}+\sum_{\substack{j\neq i\\ j\in\mathbf{m}}}\Big(\sum_{s\in\mathbf{q}_j}\mathscr{x}_{js}\Big)\Big) \tag{A 1}$$

Proof

From the relation between $\mathscr{R}_i$ and $\mathscr{V}_i$ for $i\in\mathbf{m}$ the condition

$$\mathscr{B}=\sum_{i\in\mathbf{m}}(\mathscr{B}\cap\mathscr{R}_i)$$

is equivalent to

$$\mathscr{B}=\sum_{i\in\mathbf{m}}(\mathscr{B}\cap\mathscr{V}_i) \tag{A 2}$$

It is not difficult to show (Warren 1974 a, § 4.3) that given the definitions of $\mathscr{K}_i$ and $\mathscr{N}_i$, $i\in\mathbf{m}$

$$\mathscr{V}_i\subset\bigcap_{\substack{j\neq i\\ j\in\mathbf{m}}}\Big(\bigcap_{s\in\mathbf{q}_i}\mathscr{V}_{js}\Big),\quad i\in\mathbf{m}$$

whence (A 2) becomes

$$\mathscr{B}=\sum_{i\in\mathbf{m}}\Big(\mathscr{B}\cap\Big(\bigcap_{\substack{j\neq i\\ j\in\mathbf{m}}}\Big(\bigcap_{s\in\mathbf{q}_i}\mathscr{V}_{js}\Big)\Big)\Big) \tag{A 3}$$

where equality holds as one inclusion is obvious. Taking complements of (A 3) yields

$$\mathscr{N}=\bigcap_{i\in\mathbf{m}}\Big(\mathscr{N}+\sum_{\substack{j\neq i\\ j\in\mathbf{m}}}\Big(\sum_{s\in\mathbf{q}_i}\mathscr{V}_{js}^{\perp}\Big)\Big) \tag{A 4}$$

But since $\mathscr{V}_{js}^{\perp}=\mathscr{S}_{js}+\mathscr{x}_{js}$ for $s\in\mathbf{q}_i$ and $j\in\mathbf{m}$, (A 1) follows immediately from (A 4). (q.e.d.)

At this point we wish to emphasize that the converse of Lemma A 1 is not generally true. That is, (A 3) need not imply (A 2). Of course, if $\mathscr{V}_i=\bigcap_{\substack{j\neq i\\ j\in\mathbf{m}}}\Big(\bigcap_{s\in\mathbf{q}_i}\mathscr{V}_{js}\Big)$ for all $i\in\mathbf{m}$, then the converse would hold, as would be the case if the suboutputs were all scalars, i.e. we were considering Morgan's Problem.

To simplify notation in (A 4) let us define

$$\mathscr{Z}_j=\sum_{s\in\mathbf{q}_j}\mathscr{x}_{js},\quad j\in\mathbf{m}$$

$$\mathscr{Z}_i^*=\sum_{\substack{j\neq i\\ j\in\mathbf{m}}}\mathscr{Z}_j,\quad i\in\mathbf{m}$$

whence (A 1) becomes

$$\mathscr{N}=\bigcap_{i\in\mathbf{m}}(\mathscr{N}+\mathscr{Z}_i^*) \tag{A 5}$$

Then the next step in our development is a technical, but relatively straightforward lemma which reduces (A 5) to a series of simpler subspace inclusions.

Lemma A 2

Condition (A 5) is true if and only if

$$\mathscr{Z}_i \cap (\mathscr{N} + \mathscr{Z}_i^*) \subset \mathscr{N}, \quad i \in \mathbf{m} \tag{A 6}$$

Proof

(Necessity.) Assume to the contrary that there exists an $x \in \mathscr{Z}_i$ such that $x \in \mathscr{N} + \mathscr{Z}_i^*$, and $x \notin \mathscr{N}$. But since $\mathscr{Z}_i \subset \mathscr{Z}_j^*$ for $j \neq i$, with $i, j \in \mathbf{m}$

$$x \in (\mathscr{N} + \mathscr{Z}_i^*) \cap (\bigcap_{j \neq i} (\mathscr{N} + \mathscr{Z}_j^*) = \mathscr{N}$$

by (A 5), a contradiction.

(Sufficiency.) Assuming (A 6) is true, choose x such that

$$x \in \bigcap_{i \in \mathbf{m}} (\mathscr{N} + \mathscr{Z}_i^*)$$

Therefore

$$x = w_i + \sum_{\substack{j \neq i \\ j \in \mathbf{m}}} z_{ij}, \quad i \in \mathbf{m}$$

where $w_i \in \mathscr{N}$ and $z_{ij} \in \mathscr{Z}_j$, $i, j \in \mathbf{m}$. In particular

$$x = w_1 + z_{12} + \ldots + z_{1m} = w_2 + z_{21} + z_{23} + \ldots + z_{2m}$$

Thus it follows that $z_{12} \in (\mathscr{N} + \mathscr{Z}_2^*) \cap \mathscr{Z}_2$, implying that $z_{12} \in \mathscr{N}$ by (A 6). Similarly, we may show $z_{1s} \in \mathscr{N}$ for $s \neq 1$, $s \in \mathbf{m}$, which implies $x \in \mathscr{N}$, proving the result. (q.e.d.)

The next step in our development consists of showing that (A 6) may be reduced to a series of statements about the one-dimensional subspace $\mathscr{z}_{js}$ for $s \in \mathbf{q}_j$, and $j \in \mathbf{m}$. Indeed we will show that (A 6) is equivalent to

$$\mathscr{z}_{js} \cap (\mathscr{N} + \mathscr{Z}_j^*) = 0 \quad \text{for } s \in \mathbf{q}_j, \text{ and } j \in \mathbf{m} \tag{A 7}$$

We note that (A 7) does not follow immediately from (A 6) for arbitrary subspaces; it will be necessary to exploit the particular structure of the $\mathscr{z}_{js}$ to arrive at the desired conclusion.

Lemma A 3

Condition (A 6) is true if and only if (A 7) holds,

$$\mathscr{z}_{js} \cap (\mathscr{N} + \mathscr{Z}_j^*) = 0, \quad \text{for } s \in \mathbf{q}_j,\ j \in \mathbf{m}$$

Proof

(Necessity.) By the definition of $\mathscr{z}_{js}$,

$$\mathscr{z}_{js} \cap \mathscr{N} = 0 \quad \text{for } s \in \mathbf{q}_j,\ j \in \mathbf{m}$$

Assuming (A 6) holds we have by the previous lemma

$$\mathscr{N} = \bigcap_{i \in \mathbf{m}} (\mathscr{N} + \mathscr{Z}_i^*)$$

whence

$$\mathscr{z}_{js} \cap \mathscr{N} = \mathscr{z}_{js} \cap (\bigcap_{\substack{i \neq j \\ i \in \mathbf{m}}} (\mathscr{N} + \mathscr{Z}_i^*)) \cap (\mathscr{N} + \mathscr{Z}_j^*) = 0, \quad \text{for } s \in \mathbf{q}_j,\ i \in \mathbf{m}$$

But since $\mathscr{z}_{js} \subset \mathscr{Z}_i^*$ for $j \neq i$, $i, j \in \mathbf{m}$, it follows that

$$\mathscr{z}_{js} \cap (\mathscr{N} + \mathscr{Z}_j^*) = 0, \quad \text{for } s \in \mathbf{q}_j,\ j \in \mathbf{m}$$

(Sufficiency.) Consider the subspaces $\mathscr{N} + \mathscr{Z}_i^*$, $i \in \mathbf{m}$. Since rank $B = m$ we have dim $\mathscr{N} = n - m$. Furthermore, it follows from (A 7) that $\mathscr{Z}_i^*$, for each $i \in \mathbf{m}$, contains at least $m-1$ independent vectors which are also independent of $\mathscr{N}$, and hence dim $(\mathscr{N} + \mathscr{Z}_i^*) \geqslant n-1$, $i \in \mathbf{m}$.

To demonstrate this fact let us fix $i = 1$, and choose the set of $m-1$ vectors in $\mathscr{Z}_1^*$,

$$\mathscr{z}_{j1} = (A')^{d_{j1}} C_{j1}' \in \mathscr{Z}_j, \quad \text{for } j \neq 1,\ j \in \mathbf{m}$$

We note that $\mathscr{z}_{j1} = \text{Span}\,\{z_{j1}\}$, $j \in \mathbf{m}$. Choose a basis $w_1, \ldots, w_{n-m}$ of $\mathscr{N}$. Since $\mathscr{z}_{21} \cap \mathscr{N} = 0$, it follows that the vectors $\{z_{21}, w_1, \ldots, w_{n-m}\}$ are independent. Then by (A 7), $\mathscr{z}_{31} \cap (\mathscr{N} + \mathscr{Z}_3^*) = 0$, implying $\mathscr{z}_{31} \cap (\mathscr{N} + \mathscr{Z}_2) = 0$ whence the vectors $\{z_{21}, z_{31}, w_1, \ldots, w_{n-m}\}$ are independent. Continuing (A 7) implies $\mathscr{z}_{41} \cap (\mathscr{N} + \mathscr{Z}_2 + \mathscr{Z}_3) = 0$ and hence the vectors $\{z_{21}, z_{31}, \mathscr{z}_{41}, w_1, \ldots, w_{n-m}\}$. By repeated application of (A 7) we achieve an independent set of $n-1$ vectors in $\mathscr{N} + \mathscr{Z}_1^*$, $\{z_{21}, \ldots, z_{m1}, w_1, \ldots, w_{n-m}\}$. Using a similar construction we may show an identical result for any $i \in \mathbf{m}$, i.e. dim $(\mathscr{N} + \mathscr{Z}_i^*) \geqslant n-1$. Appealing once more to (A 7), we see that we must have strict equality in the result above, hence

$$\dim\,(\mathscr{N} + \mathscr{Z}_i^*) = n - 1, \quad i \in \mathbf{m} \tag{A 8}$$

Now

$$\dim\,(\mathscr{Z}_i \cap (\mathscr{N} + \mathscr{Z}_i^*)) = \dim\,\mathscr{Z}_i + \dim\,(\mathscr{N} + \mathscr{Z}_i^*) - \dim\,(\mathscr{N} + \sum_{j \in \mathbf{m}} \mathscr{Z}_j),\ i \in \mathbf{m}$$

From (A 7) and (A 8) it follows that

$$\dim\,(\mathscr{N} + \sum_{j \in \mathbf{m}} \mathscr{Z}_j) = n$$

whence

$$\dim\,(\mathscr{Z}_i \cap (\mathscr{N} + \mathscr{Z}_i^*)) = \dim\,\mathscr{Z}_i - 1, \quad i \in \mathbf{m} \tag{A 9}$$

Consider now the subspaces $\mathscr{N} + \mathscr{Z}_i$, $i \in \mathbf{m}$. Since $\mathscr{z}_{i1} \in \mathscr{Z}_i$, and $\mathscr{z}_{i1} \cap \mathscr{N} = 0$, it follows that

$$\dim\,(\mathscr{N} + \mathscr{Z}_i) \geqslant n - m + 1, \quad i \in \mathbf{m} \tag{A 10}$$

For concreteness, choose $i = 1$. Then by (A 7) there exists $\mathscr{z}_{21} \in \mathscr{Z}_2$ such that $\mathscr{z}_{21} \cap (\mathscr{N} + \mathscr{Z}_1) = 0$ and thus

$$\dim\,(\mathscr{N} + \mathscr{Z}_1 + \mathscr{Z}_2) - \dim\,(\mathscr{N} + \mathscr{Z}_1) \geqslant 1$$

Continuing it follows that

$$\dim\,(\mathscr{N} + \mathscr{Z}_1 + \mathscr{Z}_2 + \mathscr{Z}_3) - \dim\,(\mathscr{N} + \mathscr{Z}_1 + \mathscr{Z}_2) \geqslant 1$$

whence

$$\dim\,(\mathscr{N} + \mathscr{Z}_1 + \mathscr{Z}_2 + \mathscr{Z}_3) - \dim\,(\mathscr{N} + \mathscr{Z}_1) \geqslant 2$$

Applying (A 7) repeatedly, we may readily establish

$$\dim\,(\mathscr{N} + \sum_{j \in \mathbf{m}} \mathscr{Z}_j) - \dim\,(\mathscr{N} + \mathscr{Z}_1) \geqslant m - 1 \tag{A 11}$$

Comparing (A 10) for $i=1$, and (A 11) it is immediately clear that strict equality must hold in each. Since the construction yielding (A 11) is valid for $i \neq 1$, $i \in \mathbf{m}$, it follows that

$$\dim(\mathcal{N}+\mathcal{X}_i)=n-m+1, \quad i\in\mathbf{m} \tag{A 12}$$

Therefore, from (A 12) we have

$$\dim(\mathcal{X}_i \cap \mathcal{N})=\dim \mathcal{X}_i+\dim \mathcal{N}-\dim(\mathcal{X}_i+\mathcal{N})=\dim \mathcal{X}_i-1, i\in\mathbf{m} \tag{A 13}$$

Comparing (A 9) with (A 13) and noting

$$\mathcal{X}_i \cap \mathcal{N} \subset \mathcal{X}_i \cap (\mathcal{N}+\mathcal{X}_i^*), \quad i\in\mathbf{m}$$

it is immediate that

$$\mathcal{X}_i \cap (\mathcal{N}+\mathcal{X}_i^*)=\mathcal{X}_i \cap \mathcal{N} \subset \mathcal{N}$$

which was to be proved. (q.e.d.)

Now we are ready to demonstrate the final step in our development. First we define the augmented D matrix for this type of system as in § 4 and then show that (A 7) is equivalent to statements about the ranks of submatrices of this matrix.

Lemma A 4

The condition (A 7),

$$x_{js} \cap (\mathcal{N}+\mathcal{X}_j^*)=0, \quad \text{for } s\in\mathbf{q}_j, \text{ and } j\in\mathbf{m}$$

is true if and only if :

(i) for every set $(s_1, \ldots, s_m)$ with $s_j \in \mathbf{q}_j$, $j\in\mathbf{m}$, the submatrix $D_{(s_1, \ldots, s_m)}$ has rank m ;

(ii) for every $i\in\mathbf{m}$, the submatrix D_i^* has rank $m-1$.

Proof

(Necessity.) For any set $(s_1, \ldots, s_m)$ with $s_j \in \mathbf{q}_j$, $j\in\mathbf{m}$, choose the set of m vectors $\{z_{1s_1}, \ldots, z_{ms_m}\}$, $z_{js_j}=(A')^{d_{js_j}}C_{js_j}'$. From (A 7) we have

$$x_{js_j} \cap (\mathcal{N}+\mathcal{X}_j^*)=0 \quad \text{for } s_j\in\mathbf{q}_j,\ j\in\mathbf{m}$$

where of course $x_{js_j}=\text{Span}\{z_{js_j}\}$. Hence by a construction similar to that used to show $\dim(\mathcal{N}+\mathcal{X}_i^*)=n-1$, (A 8), we may show

$$\dim(x_{1s_1}+\ldots+x_{ms_m}+\mathcal{N})=n \tag{A 14}$$

Indeed

$$\dim(x_{1s_1}+\ldots+x_{ms_m}+\mathcal{N})=\dim x_{1s_1}+\dim(x_{2s_2}+\ldots+x_{ms_m}+\mathcal{N}) - \dim(x_{1s_1}\cap(x_{2s_2}+\ldots+x_{ms_m}+\mathcal{N}))$$

but the last term is zero by (A 7). Continuing in a like manner (A 14) is established.

Now from (A 14) and the fact $\mathcal{N}=\text{Ker } B'$,

$$\dim \mathcal{B}'=\dim(B'(x_{1s_1}+\ldots+x_{ms_m}))=m$$

which of course is equivalent to

$$\operatorname{rank}\,(B'[z_{1s_1}\,;\ \dots\,;\ z_{ms_m}])=m \tag{A 15}$$

But $B'[z_{1s_1}\,;\ \dots\,;\ z_{ms_m}]=D'_{(s_1,\ \dots,\ s_m)}$, and thus (A 15) implies i).

From (A 8) we have $\dim\,(\mathscr{N}+\mathscr{Z}_i^*)=n-1$, $i\in\mathbf{m}$, whence

$$\dim\, B'(\mathscr{N}+\mathscr{Z}_i^*)=\dim\, B'\mathscr{Z}_i^*=\operatorname{rank}\,(D_i^*)'\leqslant m-1, \quad i\in\mathbf{m}$$

But from (A 15)

$$\operatorname{rank}\,(B'[z_{1s_1}\,;\ \dots\,;\ z_{i-1,\,s_{i-1}}\,;\ z_{i+1,\,s_{i+1}}\,;\ \dots\,;\ z_{ms_m}])=m-1,\ i\in\mathbf{m} \tag{A 16}$$

Since the matrix in (A 16) is a submatrix of $(D_i^*)'$, the desired result,

$$\operatorname{rank}\,(D_i^*)'=\operatorname{rank}\,D_i^*=m-1, \quad i\in\mathbf{m}$$

follows.

(Sufficiency.) If $D_{(s_1,\ \dots,\ s_m)}$ is of full rank for all sets $(s_1, \dots, s_m)$ with $s_j\in\mathbf{q}_j$, $j\in\mathbf{m}$, then (A 14) holds, i.e.

$$\dim\,(z_{1s_1}+\dots+z_{ms_m}+\mathscr{N})=n$$

For $D'_{(s_1,\ \dots,\ s_m)}=B'[z_{1s_1}\,;\ \dots\,;\ z_{ms_m}]$, hence the vectors $B'z_{1s_1}, \dots, B'z_{ms_m}$ must be independent. But this implies $z_{1s_1}, \dots, z_{ms_m}$ are independent of $\mathscr{N}=\operatorname{Ker} B'$, and thus (A 14) holds. Also

$$\operatorname{rank}\,D_i^*=\operatorname{rank}\,(D_i^*)'=\dim\,(B'\mathscr{Z}_i^*)=m-1, \quad i\in\mathbf{m}$$

Now from (A 14) we have

$$\dim\,(\mathscr{N}+\mathscr{Z}_i^*)\geqslant n-1, \quad i\in\mathbf{m}$$

But since

$$\dim\,(B'(\mathscr{N}+\mathscr{Z}_i^*))=\dim\,(B'\mathscr{Z}_i^*)<\dim\,\mathscr{B}', \quad i\in\mathbf{m}$$

it follows that $\mathscr{N}+\mathscr{Z}_i^*\neq R^n$, and thus

$$\dim\,(\mathscr{N}+\mathscr{Z}_i^*)=n-1, \quad i\in\mathbf{m}$$

As

$$z_{is_i}+\sum_{\substack{j\neq i\\ j\in\mathbf{m}}} z_{js_j}+\mathscr{N}\subset z_{is_i}+\mathscr{Z}_i^*+\mathscr{N}, \quad i\in\mathbf{m}$$

(A 14) implies

$$\dim\,(z_{is_i}+\mathscr{Z}_i^*+\mathscr{N})=n, \quad i\in\mathbf{m}$$

Then we have

$$\dim\,(z_{is_i}\cap(\mathscr{N}+\mathscr{Z}_i^*))=\dim\,z_{is_i}+\dim\,(\mathscr{N}+\mathscr{Z}_i^*)-\dim\,(z_{is_i}+\mathscr{N}+\mathscr{Z}_i^*)$$

$$=1+n-1-n=0 \quad \text{for } s_i\in\mathbf{q}_i,\ i\in\mathbf{m}$$

which establishes (A 7) and the proposition. (q.e.d.)

Theorem 1 now follows directly from the Wonham–Morse result (6) and the preceding four lemmas.

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